On computing dominant and absorbent kernels in bipolar valued digraphs

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Abstract

In this communication, we introduce an original algorithm for computing both dominant and absorbent kernels in a bipolar ordinal valued digraph. The approach relies on theoretical results recently obtained by Bisdorff, Pirlot and Roubens [8] which give a constructive – fixpoint equation based – proof of the bijection between bipolar ordinal valued kernels of such a valued digraph and the crisp kernels we observe in the associated strict median cut crisp digraph.

Keywords: Bipolar valued digraphs, Kernels, fixpoint equation

1 Dominant and absorbent kernels in digraphs

In this section we discuss the concept of independent dominant, respectively absorbent, choice in a digraph. These concepts correspond to the notion of independent dominating sets in undirected graphs. In digraphs, we distinguish two oriented variants of this notion, a dominant kernel, that is an initial, or dominant choice, and a terminal or absorbent choice.

In the graph theory literatures, dominating sets in general, and independent dominating sets in particular, have gained more and more attention (see Haynes, Hedetniemi and Slater [13]). The oriented counterpart, known as kernels in the absorbent version, and (Von Neumann game) solutions in the dominant version, have attracted slightly less attention (Ghoshal, Laskar, and Pillone [12]).

In this first part, we first introduce basic notation in the crisp case, before extending our presentation to bipolar ordinal valued digraphs.

1.1 The crisp case

We consider $G(X, R)$ to be a digraph where X is a finite set of vertices and R is a set of directed arcs, in fact a binary relation on X, i.e. $R \subseteq X \times X$. We thus ignore multiple loops and multiple arcs except for pairs of opposite arcs. If $R = X \times X$ we call G complete. The cardinality n of X gives the order of the G, whereas the cardinality of R over the square n^2 of the order of the graph gives the fill rate of the graph. We call G a connected digraph if the symmetric and transitive closure of G corresponds to a complete graph. In fact a connected graph is a graph that contains no isolated vertices. A graph $G(X, R)$ is called irreflexive if $(x, x) \notin R : \forall x \in X$. In the sequel we shall only consider irreflexive and connected digraphs.

A choice Y in a digraph $G(X, R)$ is a non empty subset of vertices from X. A dominant (respectively absorbent) choice in G is either X, called the greedy choice, or a choice $Y \subset X$ such that $\forall x \in X - Y$ there exists some $y \in Y$ such that (y, x) (respectively (x, y) is in R. An independent choice is either a singleton choice, or a choice Y such that $(x, y) \notin R$ for all $x, y \in Y$. We call dominant (respectively absorbent) kernel of G a choice Y which is both independent and dominant (respectively absorbent).

Let $G(X, R)$ be a digraph. We may represent choices with the help of a bi-valued characteristic (row) vector $Y() : X \to \{-1, 1\}$ where $Y(x) = 1$ if $x \in Y$ and $Y(x) = -1$ if $x \notin X$.

We may also represent the binary relation R with the help of the same bi-valued characteristic matrix $R() : X \times X \to \{-1, 1\}$ where $R(x, y) = 1$ if $(x, y \in R$ and $R(x, y) = -1$ if $(x, y) \notin R$.

Proposition 1 (Berge 1958)
Let $G(X, R)$ be an irreflexive digraph. A vector Y satisfying the following equation system:

$$Y \circ R \ = \ -Y \qquad (1)$$

where, for all $x \in X$,

$$(Y \circ R)(x) \ = \ \max_{y \in X} \big(\min(Y(x), R(y, x)) \big),$$

characterizes an absorbent kernel of G.

A same proposition is true for the dominant case. We simply have to reverse the direction of relation R by using instead the transposed characteristic matrix R^t of R.

Equation system (1) (respectively its reversed version) is called the absorbent (respectively dominant) kernel equation system. We don't have the space here to give exhaustive results on existence, uniqueness and particular characteristics of the solutions of equation system (1) (see Berge [1, 2] and Ghoshal et al. [12] for instance). Quickly stated, acyclic digraphs always admit a unique dominant and a unique absorbent kernel ([1]). Almost all random digraphs have several dominant as well as absorbent kernels ([11, 18]). But asymmetric odd circuits for instance don't admit any dominant nor absorbent kernel ([14]).

The digraphs we would like to consider here appear in the context of the outranking methods designed for multiple criteria decision aiding (Roy and Bouyssou [15]). These outranking digraphs are generally valued with the help of a bipolar majority concordance based index (Bisdorff [6, 7]).

1.2 The bipolar ordinal valued case

Instead of characterizing membership statements with a simple bi-valued $\{-1, 1\}$ characteristic function, we consider here a finite increasing sequence L of degrees of credibility denoted $(-m, -(m-1), \ldots, -1, 0, +1, \ldots, +m)$. As in the bi-valued case, the sign of the characteristic value is carrying a logical denotation: $+$ signifies *more true than false* and $-$ signifies *more false than true*. When considering two degrees of credibility k and l such that $k > l$ the membership assertion evaluated k is considered to be strictly more credible than the one evaluated l; with $-m$ meaning definitely false and $+m$ meaning definitely true. 0 signifies that the so characterized membership assertion is neither true nor false, i.e. logically undetermined (see Bisdorff [7]).

Let us consider a crisp choice Y. We denote its corresponding L-valued characteristic function $\tilde{Y} : X \to L$, where $\tilde{Y}(x) = \pm k$ gives the degree of truthfulness ($+k$) or falsity ($-k$) of the fact that x is a member of the choice Y. The L-valued characteristic (row) vector $\tilde{Y}$ is called an L-choice if $\tilde{Y}$ does not contain any 0 value.

Similarly, for a given relation $R \subseteq X \times X$ we may as well consider such an L-valued characteristic matrix $\tilde{R} : X \times X \to L$ where $\tilde{R}(x, y) = \pm k$ gives the degree of truthfulness ($+k$) or falsity ($-k$) of assertion $(x, y) \in R$. The corresponding L-valued digraph is denoted $\tilde{G}(X, \tilde{R})$.

To each L-valued digraph $\tilde{G}(X, \tilde{R})$, we may associate a crisp digraph $G(X, R)$ obtained by operating a strict median cut on $\tilde{R}$ as follows: $R = \{(x, y) \in X \times X / \tilde{R}(x, y) > 0\}$. Similarly, to each L-choice $\tilde{Y}$ we may associate a natural choice Y obtained by the same strict median cut: $Y = \{x \in X / \tilde{Y} > 0\}$.

Furthermore, when considering two L-valued characteristic vectors $\tilde{Y}_1$ and $\tilde{Y}_2$, we may consider a special bipolar sharpness relation $\preceq$ defined as follows: $\tilde{Y}_1 \preceq \tilde{Y}_2$ if $\forall x \in X$, either $\tilde{Y}_1(x) \geq \tilde{Y}_2(x) \geq 0$ or $\tilde{Y}_1(x) \leq \tilde{Y}_2(x) \leq 0$.

We may now consider the L-valued extension of the crisp kernel equation system (1) where the get the following formal result:

Proposition 2 *Let $\tilde{G}(X, \tilde{R})$ be an L-valued digraph. The choices Y obtained from the maximal sharp L-choices $\tilde{Y}$ – in the sense of relation $\preceq$ – which verify the following L-valued kernel equation system:*

$$\tilde{Y} \circ \tilde{R} \ = \ -\tilde{Y} \qquad (2)$$

where

$$(\tilde{Y} \circ \tilde{R})(x) \ = \ \max_{y \in X, y \neq x} \big(\min(\tilde{Y}(x), \tilde{R}(x, y)) \big),$$

for all $x \in X$, are the dominant kernels Y of the associated strict median cut digraph $G(X,R)$.

Proof of this important and non trivial proposition (Bisdorff, Pirlot and Roubens [8]) is based on an original constructive fixpoint equation based approach, imagined by Pirlot, which allows to precisely construct the maximal sharp L-choices that are solutions of the L-valued kernel equation system (2). He observed that, given a dominant kernel Y of the associated strict median cut graph G, the fixpoint of the transformation:

$$\mathcal{T}(\tilde{Y}) = -(\tilde{Y} \circ \tilde{R}) = \tilde{Y}, \tag{3}$$

which necessarily exists and is unique, gives the corresponding maximal sharp $\tilde{Y}$ solution of kernel equation system (2).

By simply transposing the L-valued relation $\tilde{R}$ in this equation system we get a similar bijection for the absorbent kernel case.

The fixpoint equation (3) based proof of proposition (2) gives by the way a constructive procedure for effectively computing L-valued dominant and absorbent kernels in an L-valued digraph.

2 The L-valued kernel extraction

In this part, we present an original algorithm for computing bipolar ordinal valued kernels from a given digraph $\tilde{G}$.

2.1 The general algorithm

Von Neumann [19] showed that the unique dominant kernel Y of a crisp acyclic digraph $G(X,R)$ corresponds to the stable solution of a dual fixpoint equation (see Schmidt and Ströhlein [16, 17]):

$$\mathcal{T}^2(Y) = -(-(Y \circ R) \circ R) = Y. \tag{4}$$

Solving equation (4) allows to compute the kernel Y in polynomial complexity in terms of the order of G.

Based on this result, Bisdorff [5] observed that for a given L-valued digraph $\tilde{G}$, each independent and dominant choice Y in the associated strict median cut graph G determines a partially defined subgraph $\tilde{G}_{/Y}$ which admits a corresponding L-choice $\tilde{Y}$ as unique maximal sharp solution of kernel equation system (2). Following the von Neumann approach, a similar dual L-valued extended fixpoint algorithm applied to $\tilde{G}_Y$ allows to compute the associated maximal sharp $\tilde{Y}$ solution in polynomial complexity.

As already mentioned, the fixpoint equation (3) now gives us access to a similar algorithmic approach. For each kernel Y observed in the associated strict median cut digraph G, we may compute the associated maximal sharp L-choice $\tilde{Y}$ by solving the corresponding fixpoint equation $\mathcal{T}(\tilde{Y}) = \tilde{Y}$, where $\tilde{Y}$ is initialized as $\{-m, +m\}$-valued characteristic vector of kernel Y.

In general, for a given L-valued digraph $\tilde{G}$:

1. we compute the associated strict median cut digraph G;
2. we compute the sets $\mathcal{K}^d$ (respectively $\mathcal{K}^a$) of dominant (respectively absorbent) absorbent kernels in G;
3. for each kernel $Y \in \mathcal{K}^d$ (respectively $\mathcal{K}^a$), we solve the fixpoint equation $\mathcal{T}(\tilde{Y}) = \tilde{Y}$.

Step (1) and (3) are of polynomial complexity in terms of the order the digraph and the number m of truthfulness degrees, but step (2) remains evidently difficult, as finding a kernel in a general digraph, via a reduction from the SAT problem, has been showed to be NP-complete by Chvatál (http://www.cs.rutgers.edu/ chvatal/kernel.html).

2.2 On computing crisp kernels from a crisp digraph

Computing dominant and absorbent kernels in a crisp digraph is indeed a computationally difficult problem because of the fact that the crisp kernel equation system (1) does not allow in general any triangular solving approach. Therefore only the smart enumeration of potential choices remains an efficient approach.

In Bisdorff [3, 4, 5], we used a finite domain solver to generate all admissible solutions of the crisp kernel equation system (1), but working in a more general tri-valued $\{-1, 0, +1\}$ evaluation domain. The constraint enumerated solutions are then sorted with respect to the sharpness relation $\preceq$ in order to get the maximal sharpest solutions. Thus we may obtain not only L-choices, i.e. completely L-determined solutions, but also only partially L-determined solutions which have no correspondent in the associated crisp strict median cut digraph G. Unfortunately, finite domain solvers, like the one proposed in GNU-Prolog (see [9, 10]), due to space limitation when generating the constraints graph used for efficient arc-consistency propagation, may not easily tackle graphs of orders larger than 100. Also, the set of admissible solutions of the kernel system (1) with graphs of large order may be of huge cardinality. We therefore looked for a smart enumeration technique with ad hoc propagation and search space cutting strategies.

Noticing that in the outranking methods, the strict median cut digraphs obtained from the given L-valued digraphs have in general a high fill rate (50% and more), we observe most of the time a small number of kernels, i.e. independent and dominant or absorbent choices, that are furthermore of small cardinality compared to the order of graph (see [18]).

Therefore, starting from each singleton choice, our approach consists in constructing larger and larger independent choices until we reach one that is minimally dominant or absorbent. Indeed, a dominant (respectively absorbent kernel) is necessarily both a maximally independent and minimally dominant (respectively absorbent) choice (see Berge [2]). Keeping by the way record of already visited choices, we avoid unnecessary repetitions when progressing with the exploration of admissible independent choices.

The great advantage of this algorithm, directly working on the enumeration of independent choices is that we may in the same enumeration pick up both the dominant and absorbent kernels. Enumerating the solutions of the kernel equation system (1) needs separate runs for the dominant and the absorbent case.

2.3 Implementation and empirical run tests

We have implemented the complete kernel extraction algorithm with all three steps in Python (version 2.4), using the latest inbuilt set class with optimized set manipulating operators on a four processors i64 architecture under GNU/Linux 2.4. Run time statistics shown in Figure 1 illustrate that we are able to extract in a fraction of a second all L-valued dominant and absorbent kernels of randomly filled L-valued digraphs of order up to 60 and fill rate over 25 %. With an average

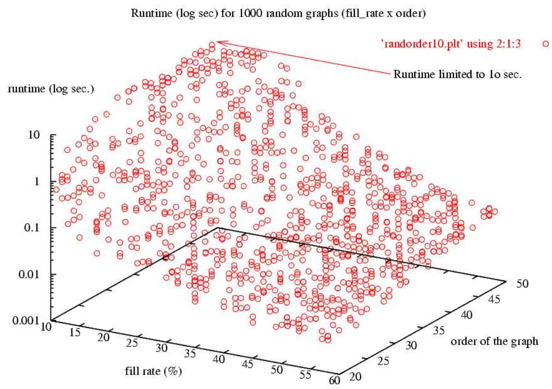

Figure 1: Timing the kernels extraction algorithm

high fill rate of 75% and more, we are even able to extract all dominant and absorbent kernels in random digraphs of order up to 700 in less than 10 seconds. In this case, kernels are indeed of very low cardinality, in general less than 6 (see [18]).

Worst case is given with sparse connected digraphs of very low fill rate, where the kernel cardinalities may become high – up to half of the order of the graph – and the search space, despite our cut mechanisms, gets definitely to large and therefore inexplorable. Figure 1 shows this exponential growing in the North East corner where the yrun times are deliberately limited to 10 seconds. With an order of 30 and a fill rate lower than 10% we may for instance observe run times of nearly 100 seconds, compared to the otherwise very low times of less than a second. We may see that here the combinatorial explosion is brutal and our algorithm is not adapted for digraphs with such low a fill rate.

3 Concluding remarks

In this communication, we present an original algorithm for computing kernels in a bipolar valued digraph. Empirical run test show efficient execution times for digraphs of sufficient fill rate (50% and more).

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