Random Outranking Digraphs

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Motivation

- Provide instances of genuine valued outranking digraphs (VODs) for MCDA method debugging
- Discover VOD's specific structural characteristics
- Comparison with other kinds of random valued digraphs
- Mathematical characterisation of VODs

Valued Outranking Digraphs

Fact: A crisp outranking digraph may model any reflexive relation, but not every valued digraph is a valid instance of a VOD !

Example (bipolar valued digraphs)

A valid VOD instance

$\widetilde{S}$	a01	a02	a03	a04
a01	-	0.2	0.4	0.4
a02	0.0	-	0.2	0.2
a03	0.4	0.6	-	0.0
a04	-0.2	0.2	0.2	-

An invalid VOD instance

$\widetilde{S}$	a01	a02	a03	a04
a01	-	0.2	0.4	-0.3
a02	0.0	-	0.2	0.2
a03	0.4	0.6	-	0.0
a04	-0.2	0.2	0.2	-

$$\boxed{-1.0 < (x\widetilde{S}y) < 0.0 \quad \Rightarrow \quad (y\widetilde{S}x) \geqslant 0.0\,, \quad \forall x, y}$$

valid VODs

Comments

A performance tableau shows the performances of a finite set X of decision actions on a finite set F of criteria-functions associated with significance weights and discrimination thresholds.

A valued digraph is a valid VOD iff there exists a performance tableau which generates the apparent digraph valuation.

Outline

The outranking situation

- Let X be a finite set of p alternatives.
- Let F be a finite set of $n > 1$ criteria.
- Let m be the total significance of the criteria.
- Let x and y be two alternatives from X.
- Let x_i be the value taken by x on criterion g_i

Definition (The outranking situation)

- x outranks y ($x \,\mathrm{S}\, y$) if there is a significant majority of criteria which support an at least as good statement and there is no criterion which raises a veto against it.
- The bipolar valued relation $\widetilde{\mathrm{S}} \in [-m, m]$ expresses the significance of the validation or the non-validation of the outranking relation S.

General VOD definition – continued

Definition (The bipolar valued outranking situation)

$$\widetilde{\mathrm{S}}(x,y) = \min\left\{ \left(\sum_{i\in F} w_i \cdot C_i(x,y)\right), \min_{i\in F}\left(-V_i(x,y)\right)\cdot m \right\}$$

$$C_i(x,y) = \begin{cases} 1 & \text{if} \quad x_i + q_i > y_i; \\ -1 & \text{if} \quad x_i + p_i \leqslant y_i; \\ 0 & \text{otherwise} \end{cases}$$

$$V_i(x,y) = \begin{cases} 1 & \text{if} \quad x_i + v_i \leqslant y_i; \\ -1 & \text{if} \quad x_i + wv_i > y_i; \\ 0 & \text{otherwise} \end{cases}$$

where q_i, p_i represent the weak preference, resp. the preference, and wv_i, v_i, the weak veto, resp. the veto, threshold on criterion g_i.

The VOD relation $\widetilde{\mathrm{S}}$ – continued

$\widetilde{\mathrm{S}}$ is defined on a bipolar-valued credibility scale $\mathcal{L} = [-m, m]$ supporting the following semantics denotation:

- $\widetilde{\mathrm{S}}(x,y) = +m$ means that assertion $x \,\mathrm{S}\, y$ is clearly validated.
- $\widetilde{\mathrm{S}}(x,y) = -m$ means that assertion $x \,\mathrm{S}\, y$ is clearly non-validated.
- $\widetilde{\mathrm{S}}(x,y) > 0$ means that assertion $x \,\mathrm{S}\, y$ is more validated than non-validated.
- $\widetilde{\mathrm{S}}(x,y) < 0$ means that assertion $x \,\mathrm{S}\, y$ is more non-validated than validated.
- $\widetilde{\mathrm{S}}(x,y) = 0$ means that assertion $x \,\mathrm{S}\, y$ is undetermined.

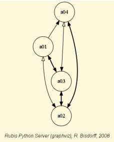
VOD $\widetilde{G}(X, \widetilde{S})$

Definition (The bipolar valued outranking digraph)

- We denote $\widetilde{G}(X, \widetilde{S})$ the bipolar-valued outranking digraph modelled via $\widetilde{S}$ on $X \times X$.
- The associated crisp outranking relation S may be recovered from $\widetilde{S}$ as the set of pairs (x, y) such that $\widetilde{S} > 0$.
- $G(X, S)$ is called the crisp outranking digraph associated with $\widetilde{G}(X, \widetilde{S})$.

A $\widetilde{G}(X, \widetilde{S})$ instance

$\widetilde{S}$	a01	a02	a03	a04
a01	-	0.2	0.4	0.4
a02	0.0	-	0.2	0.2
a03	0.4	0.6	-	0.0
a04	-0.2	0.2	0.2	-



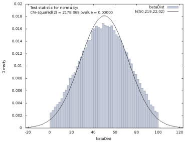
Ramdom Performance Tableau

Definition (A reference model)

- 20 decision actions; low variant: 13; high variant: 50.
- 13 criteria; low variant: 7; high variant: 20.
- All criteria are equi-significant.
- All criteria use a same cardinal scale from 0.0 to 100.0.
- Four random performance generators may be used:
  - a uniform generator ($\mathcal{U}(0.0, 100)$),
  - a truncated normal generator ($\mathcal{N}(\mu, \sigma)$),
  - a triangular generator ($\mathcal{T}(xm, r)$) with mode xm and probability repartition r,
  - a beta generator ($\mathcal{B}eta(xm, s)$) with mode xm and standard deviation s.

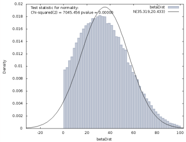
Truncated Normal Generator

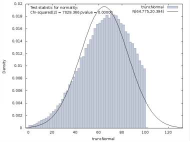
- In the reference case, the mode xm is situated in the middle (50.0) of the performance scale and the standard deviation is a fourth (25) of the scale scope.



Truncated Normal Generator (continued)

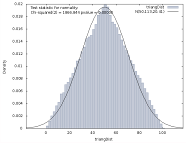
- We consider two variants:
  - low performances: $xm = 30$,
  - high performances: $xm = 70$,





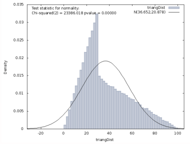
Triangular Generator

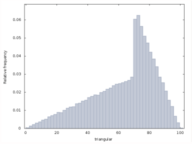
- In the reference case, the mode *xm* is situated in the middle (50.0) of the performance scale and the probabilty is equally distributed on both sides, i.e. $r = 0.5$ and *xm* represents the median performance.



Triangular Generator (continued)

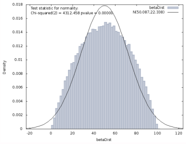
- We consider two variants with fixed repartition $r = 0.5$:
  - low performances: $xm = 30$,
  - high performances: $xm = 70$,





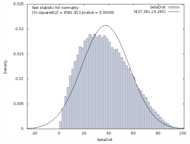
Beta Generator

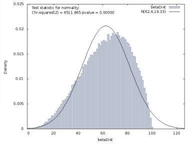
- In the reference case, the mode *xm* is situated in the middle (50.0) of the performance scale and the probabilty is equally distributed on both sides, i.e. *xm* represents the median performance.



Beta Generator (continued)

- We consider two variants with equal standard deviation:
  - low performances: $xm = 30$,
  - high performances: $xm = 70$,





Fixed Discrimination Thresholds

On each criterion, the default discrimination thresholds are chosen such that the:

- indifference threshold equals 5.0 (low: 2.5, high:10.0);
- preference threshold equals 15.0 (low: 10.0, high:20.0);
- weak veto threshold equals 70.0 (low: 60.0, high: 80);
- veto threshold equals 80.0 (low: 70.0, high: 90).

- The ordinal criteria admit solely a preference threshold of one unit.

Example

Random performance tableau instance

Fixed Percentile Discrimination Thresholds

On each criterion, the default discrimination thresholds are chosen such that the:

- indifference threshold equals the percentile 5 of all generated performance differences;
- preference threshold equals the percentile 10 of all generated performance differences;
- weak veto threshold equals the percentile 90 of all generated performance differences;
- veto threshold equals the percentile 95 of all generated performance differences.

- The ordinal criteria admit solely a preference threshold of one unit.

Example

Random performance tableau instance

Special Performance Tableaux

Definition (Ramdom Cost-Benefit Performance Tableau)

- 20 decision actions; low variant: 13; high variant: 50.
- 13 criteria; low variant: 7; high variant: 20.
- A criteria is with equal probability either to be minimized (cost criteria) or to be maximized (benefit criteria).
- All criteria either support an ordinal or a cardinal performance scale; the cost criteria being mostly cardinal (2/3) and the benefit ones mostly ordinal (2/3).
- Ordinal performances are represented on integer scales: $\{1, 2, ..., 10\}$.
- Cardinal performances are represented on a decimal scale: $[0.0; 100.0]$ with a precision of 2 digits.

Random Cost-Benefit Performance Tableau (continued)

- In the Cost-Benefit model the decision actions are divided randomly into three categories: *cheap*, *neutral*, *advantageous*.
- An action is called:
  - cheap when the performances are generated with $\mathcal{T}(xm = 30, r = 0.5)$ (reference) or $\mathcal{N}(\mu = 30, \sigma = 25)$.
  - advantageous when the performances are generated with $\mathcal{T}(xm = 70, r = 0.5)$ (reference) or $\mathcal{N}(\mu = 70, \sigma = 25)$,
  - and neutral when the performances are generated with $\mathcal{T}(xm = 50, r = 0.5)$ (reference) or $\mathcal{N}(\mu = 50, \sigma = 25)$.

Example

Random performance tableau instance

Correlating the performances with three coalitions

- In a first case we consider three a priori coalitions: A,B and C.
- Every criteria is affected randomly to one of the three coalitions.
- Each actions is randomly affected on each coalition to one of three performance following categories: low performance (-), medium performance ($\sim$) and high performance ($+$).
- When generating the performances af an alternative on a criterion, the random generator is modulated following the prerformance profile of the action respective to the coalition of the criterion.

Example

Random performance tableau instance

Random Criteria Coalitions

- We consider a family of n criteria.
- Every criteria is affected randomly to one of n potential coalitions.
- Each actions is randomly affected on each coalition to one of three performance categories: low performance (-), medium performance ($\sim$) and high performance ($+$).
- When generating the performances af an alternative on a criterion, the random generator is modulated following the prerformance profile of the action respective to the coalition of the criterion.

Example

Random performance tableau instance

Random VODs

Comments

- *From the previous reference random performance tableaux, we are going to generate a valued outranking digraph* $\widetilde{G}(X, \widetilde{S})$.
- *We call ROD a sample of 3000 such valued outranking digraphs obtained from independently sampled random performance tableaux.*
- *The sample size 3000 associates a 99% confidence to almost all average characteristics of the sample.*
- *When the standard deviation is less than* 10%, *the confidence interval around mean percentage results is thus less than* 0.5%.

First results: arc and link densities

- For the reference ROD we observe in average:
  - an arc density of 47.9%(4.4);
  - a double link density of 14.22%(4);
  - a single link density of 67.60%(6);
  - a link absence density of 18.19%(6).
- A similar sample of standard random digraphs (SRDs) with an arc probability of 48% would show a binomial probability of single links (46%), double links (25%) and absence of links (27%).
- As it is the case of SRDs, these results are in fact independent of the order of the RODs.

First results: more or less vetoes

- The number of effectively raised vetoes is directly related to the average number of cardinal criteria we observe in the ROD.

link type	20 criteria average	(stdev.)	13 criteria average	(stdev.)	7 criteria average	(stdev.)
arc	40.0%	(4.2)	47.9%	(4.4)	52.2%	(3.7)
double	9.8%	(3.0)	14.22%	(4.1)	13.9%	(4.0)
single	60.5%	(7.0)	67.60%	(6.1)	76.6%	(5.2)
absence	29.7%	(7.3)	18.19%	(6.4)	9.4%	(4.9)

- Without any vetoes, we are faced with bipolar-valued weak tournaments where there are always either a single or a double link between all pairs (x, y) of actions:

$$\boxed{(x\widetilde{S}y) + (y\widetilde{S}x) \geqslant 0.0\,, \quad \forall x, y}$$

First results: graph connectivity

- RODs, as well as SRDs always show one single component.
- SRDs nearly always (99%) show a single strong component.
- RODs, however, may show up to 8 strong components.

ROD frequency distribution of multiple strong components

nbr.	frequency	rel.	cum.	leaves
1	1210	40.33%	40.33%	**************
2	955	31.83%	72.17%	***********
3	517	17.23%	89.40%	******
4	203	6.77%	96.17%	**
5	72	2.40%	98.57%	
6	26	0.87%	99.43%	
7	13	0.43%	99.87%	
8	4	0.13%	100.00%	

First results: multiple vetoes and strong components

- Even with 20 criteria, RODs show always one single component. But, up to 11 strong components (in a digraph of order 20) may now appear:

nbr.	frequency	rel.	cum.	leaves
1	635	21.17%	21.17%	*******
2	809	26.97%	48.13%	*********
3	648	21.60%	69.73%	*******
4	423	14.10%	83.8%	*****
5	253	8.43%	92.27%	***
6	117	3.90%	96.17%	*
7	60	2.00%	98.17%	
8	28	0.93%	99.10%	
9	19	0.63%	99.73%	
10	5	0.17%	99.90%	
11	3	0.10%	100.00%	

Concluding Remarks

In this communication we have presented:

- Generators for random performance tableaux
- A reference model for random outranking digraphs
- Some empirical statistical results

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