The bipolar foundation of the outranking concept

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Abstract

In this research note, we explore the correspondence between the codual and the asymmetric part of a bipolar valued outranking relation.

Keywords: Outranking, Veto Principle, Bipolar characteristic valuation.

Contents

1 Introduction

Recently, Pirlot and Bouyssou [1] have reported that a strict (asymmetric) outranking relation defined similarly to the classic outranking [2] is in general not identical to its codual relation, that is the converse of its negation. This hiatus is problematic as the asymmetric part of an outranking relation is commonly identified as being in fact its codual relation.

In this research note we explore this problem in the context of our bipolar credibility calculus [3–5]. In a first section, following the hint of Pirlot and Bouyssou [1], we illustrate formally this unsound hiatus between the asymmetric part and the codual in the case of the classic outranking concept [6]. In a second we introduce an bipolarly extended veto principle which allows us to extend the definition of the classic outranking concept in such a way that the identity between its asymmetric part and its codual is indeed given.

2 The classic outranking concept

2.1 Overall preference aggregation

Let $A = \{x, y, z, \ldots\}$ be a finite set of potential decision alternatives and $F = \{1, \ldots, n\}$ a coherent [2] finite family of $n > 1$ criteria.

The alternatives are evaluated on each criterion $i \in F$ on a real performance scale $[0; M_i]$ supporting a constant or proportional indifference q_i and preference p_i discrimination threshold such that $0 \leqslant q_i < p_i \leqslant M_i$ [2]. The performance of alternative x on criterion i is denoted x_i.

In order to characterize a local *at least as good as* situation [4, 6] between any two alternatives x and y of A, with each criterion i is associated a double threshold order $\geqslant_i$ whose bipolar characteristic representation $r(\geqslant_i)$ is given by:

$$r(x \geqslant_i y) = \begin{cases} \ \ 1 & , \ \text{if } x_i + q_i \ \geqslant \ y_i \\ -1 & , \ \text{if } x_i + p_i \ \leqslant \ y_i \\ \ \ 0 & , \ \text{otherwise.} \end{cases} \tag{1}$$

Furthermore, we associate with each criterion $i \in F$ a rational *significance weight* w_i which represents the contribution of i to the *overall warrant or not* of the *at least as good as* preference situation between all pairs of alternatives. Let W be the set of relative significance weights associated with F such that

$$W = \{w_i : i \in F\}, \text{ with } 0 < w_i < 1 \text{ and } \sum_{i \in F} w_i = 1. \tag{2}$$

The bipolar-valued characteristic representation r of the overall *"at least as good"* relation, denoted $\geqslant$, aggregating all the partial *a*t least as good as situations $\geqslant_i$ for $i \in F$, is given by:

$$r(x \geqslant y) \ = \ \sum_{w_i \in W} w_i \cdot r(x \geqslant_i y), \tag{3}$$

For each criterion $i \in F$, we can similarly characterize a local "*b*etter than" situation between any two alternatives x and y of A with a double threshold order $>_i$ and whose bipolar numerical representation $r(>_i)$ is given by:

$$r(x >_i y) = \begin{cases} \ \ 1 & , \ \text{if } x_i - p_i \ \geqslant \ y_i \\ -1 & , \ \text{if } x_i - q_i \ \leqslant \ y_i \\ \ \ 0 & , \ \text{otherwise.} \end{cases} \tag{4}$$

Again, the overall "*better than*" is given by:

$$r(x > y) \ = \ \sum_{w_i \in W} w_i \cdot r(x >_i y), \tag{5}$$

Proposition 2.1
The asymmetric part, i.e. $(x \geqslant y)$ *and* $(y \not\geqslant x)$*, of the overall* "at least as good" *relation* $\geqslant$ *on* A *is identical to the overall* "better than" *relation* $>$ *on* A.

Proof. For each $i \in F$, $r\big((x \geqslant_i y) \wedge (y \not\geqslant_i x)\big) = r(x >_i y)$. Indeed,

$$r(\neg(y \geqslant_i x)) = \begin{cases} -1 & , \text{ if } y_i + q_i \;\geqslant\; x_i \\ 1 & , \text{ if } y_i + p_i \;\leqslant\; x_i \\ 0 & , \text{ otherwise} \end{cases} .$$

Corollary 2.2
The overall "better than" *relation* $>$ *on* A *is the codual, i.e. the converse of the negation, of the overall* "at least as good" *relation* $\geqslant$ *on* A.

Proof. The double threshold order $>_i$ on A for each criterion $i \in F$, is the codual of the double threshold order $\geqslant_i$. □

2.2 The veto principle

In order to characterize a local *veto* situation [4, 6] between any two alternatives x and y of A we may associate to each performance scale $[0; M_i]$ constant or proportional *weak veto* (wv_i) and *veto* (v_i) discrimination thresholds such that $p_i < wv_i \leqslant v_i \leqslant M_i + \epsilon$ for all i in F [6].

We may thus define on each criterion $i \in F$ a double threshold order denoted $\lll_i$ which represents a "*seriously less performing than on criterion i*" situation and whose bipolar numerical representation $r(\lll_i)$ is given by:

$$r(x \lll_i y) = \begin{cases} 1 & , \text{ if } x_i - v_i \;\leqslant\; y_i \\ -1 & , \text{ if } x_i - wv_i \;>\; y_i \\ 0 & , \text{ otherwise} \end{cases} . \qquad (6)$$

Proposition 2.3
*The local "*seriously less performing than*" relation is included in the converse of the local "*better than*" relation.*

Proof. For each $i \in F$, $r(x \lll_i y) \leqslant r(y >_i x)$. □

The bipolar characteristic representation of a "*veto*" situation [2] is now given by the overall disjunction of local "*seriously less performing than*" situations:

$$r(x \lll y) \;=\; r\Big(\bigvee_{i \in F} (x \lll_i y)\Big) \;=\; \max_{i \in F} \;\; r(x \lll_i y)\,. \qquad (7)$$

It is worthwhile noticing that:

- in case $wv_i = v_i$, we recover the classic Electre definition of the veto principle [2];
- in case $wv_i = v_i = M_i + \epsilon$, the criterion i does not support any veto principle;
- in case $wv_i < v_i = M_i + \epsilon$, the criterion only supports a *weak veto* principle.

We are now ready to define the classic outranking relation.

2.3 The classic outranking relation

The classic outranking situation is defined as follows:

Definition 2.1. *An alternative x* outranks *an alternative y, denoted $x \succcurlyeq y$, when*

1. *a* significant majority *of criteria validates the fact that x is performing at least as good as y, i.e. $x \geqslant y$,*
2. *and there is no veto raised against this validation, i.e. $x \not\lll y$.*

The corresponding bipolar numerical representation gives:

$$r(x \succcurlyeq y) \;=\; r\big(\,(x \geqslant y) \wedge (x \not\lll y)\,\big) \;=\; \min\big(\,r(x \geqslant y), r(x \not\lll y)\,\big) \tag{8}$$

Proposition 2.4 (Pirlot and Bouyssou [1])
Let $\succcurlyeq$ be a classic outranking relation.

1. *The asymmetric part $\succ$ of the classic outranking relation $\succcurlyeq$, i.e. $x \succcurlyeq y$ and $y \not\succcurlyeq x$ is in general not identical to its codual relation.*
2. *The absence of any weak or strong veto is a sufficient and necessary condition for making the asymmetric part $\succ$ of $\succcurlyeq$, i.e. $x \succcurlyeq y$ and $y \not\succcurlyeq x$ identical to the codual of $\succcurlyeq$.*
3. *The absence of any strong veto alone is not a sufficient condition for making the asymmetric part $\succ$ identical to the codual of $\succcurlyeq$.*

Proof.

(1) $r(y \;\not\succcurlyeq\; x) \;=\; \max\big(\,r(y \;\not\geqslant\; x), r(y \lll x)\,\big) \;=\; \max\big(\,r(x > y), r(y \lll x)\,\big)$ whereas $r(x \succ y) = \min\big(r(x \succcurlyeq y), r(y \;\not\succcurlyeq\; x)\big) \leqslant r(y \;\not\succcurlyeq\; x)$. The strict inequality appears when $r(y \lll x) = 1$.

(2) $wv_i = v_i = M_i + \epsilon$ implies that $r(x \succcurlyeq y) = r(x \geqslant y)$ and the claimed identity follows from Proposition 2.1. Conversely, suppose that $wv_i \leqslant v_i < M_i + \epsilon$ and there exist a strong veto situation $(r(x \lll_i y) = 1)$ on some criterion $i \in F$. In this case $\min\big(r(x \succcurlyeq y), r(y \;\not\succcurlyeq\; x)\big) \;=\; \min\big(-1, r(y \;\not\succcurlyeq\; x)\big) \;=\; -1 \;< r(y \;\not\succcurlyeq\; x) = 1$.

(3) Suppose that $wv_i \leqslant v_i = M_i + \epsilon$ and there exist a weak veto situation $r(x \lll_i y) = 0$ on some criterion $i \in F$ with $r(x > y) > 0$. The claim follows in this case from a same argumentation as under (2). $\square$

As recently reported by Pirlot and Bouyssou [1], this hiatus between the asymmetric part and the codual raises a serious concern with respect to the logical soundness of the classic outranking definition. Only the absence of any veto mechanism can guarantee this somehow necessary property from the point of view of the intended semantics of the outranking concept. But this is vanishing the very interest of the outranking concept itself.

3 Outranking with bipolar veto

3.1 The bipolar outranking concept

Let x, y be two decision alternatives. We say that x outranks y, denoted $x \widetilde{\succcurlyeq} y$, if, either, a significant majority of criteria validates a global outranking situation between x and y and no serious counter-performance is observed on a discordant criterion, or, an excellent performance is observed on at least one concordant criteria. In terms of our bipolar numeric representation r we obtain the following formal definition:

Definition 3.1 (Outranking with bipolar veto).

$$r(x \widetilde{\succcurlyeq} y) = \max\big[\min\big(r(x \geqslant y), r(x \not\lll y)\big), r(y \lll x)\big] \tag{9}$$

If $wv_i = v_i = M_i + \epsilon$ for all $i \in F$, i.e. in the absence of any vetoes, we recover the previous case where $r(x \widetilde{\succcurlyeq} y) = r(x \succcurlyeq y) = r(x \geqslant y)$. If we observe a strong veto, $r(x \lll y) = 1$ and no excellent performance for x compared to y, $r(y \lll x) = -1$, we obtain $r(x \widetilde{\succcurlyeq} y) = -1$ and $r(y \widetilde{\succcurlyeq} x) = 1$. Conversely, if we observe an excellent performance for x compared to y, $r(y \lll x) = 1$, and no veto, $r(x \lll y) = -1$, we obtain $r(x \widetilde{\succcurlyeq} y) = 1$ and $r(y \widetilde{\succcurlyeq} x) = -1$. If we observe both a veto and an excellent performance: $r(x \lll y) = 1$ and $r(y \lll x) = 1$, we get $r(x \widetilde{\succcurlyeq} y) = 1$ and $r(y \widetilde{\succcurlyeq} x) = 1$, i.e. both alternatives are considered to be *globally equivalent*.

This last result is, however, not satisfactory at all, as it implements a blind compensation of serious counter-performances on some criteria with excellent performances on others.

3.2 Doubt versus invalidation

A possible way out of this unsatisfactory situation is given by the neutral logical position of our bipolar numerical representation [3, 4]. It allows to not decide whether a statement is in fact validated or not. Instead of immediately rejecting the validation of a global outranking situation when observing a notorious counter-performance on a discordant criterion, it is more opportune to take an indeterminate position with respect to its validation or invalidation, to suspend in some way the logical assessment.

Following this idea, we are going to favour the weak veto principle by always setting the veto thresholds v_i to the ineffective value $M_i + \epsilon$ on all criteria $i \in F$. Thus a veto, the case given, may only manifest itself with an absolute weakening of the potential significance of the global outranking statement.

Proposition 3.1
When $wv_i < v_i = M_i + \epsilon$ for all $i \in F$, the bipolar outranking definition 3.1 is equivalent to the following:

$$r(x \widetilde{\succcurlyeq} y) = \max\big[\min\big(r(x \geqslant y), r(x \not\lll y)\big), \min\big(r(x \not\geqslant y), r(y \not\lll x)\big)\big]. \tag{10}$$

Proof. We have to distinguish four cases. 1. No veto and no excellent comparative performance is observed: $r(x \lll y) = -1$ and $r(y \lll x) = -1$. In this case Formula (10) is equivalent to Formula (9). 2. A weak veto and no excellent comparative performance is observed: $r(x \lll y) = 0$ and $r(y \lll x) = -1$. In this case, $r(x \widetilde{\succcurlyeq} y) = \min\big(r(x \geqslant y), 0\big)$ such that only the positive values of $r(x \geqslant y)$ are concerned with the weak veto. 3. No weak veto, but an excellent comparative performance is observed: $r(x \lll y) = -1$ and $r(y \lll x) = 0$. In this case, $r(x \widetilde{\succcurlyeq} y) = \max\big(r(x \geqslant$

$y), 0\big)$ and only the negative values of $r(x \geqslant y)$ are concerned. 4. If both a weak veto and an excellent performance are observed: $r(x \lll y) = 0$ and $r(y \lll x) = 0$, $r(x \widetilde{\succcurlyeq} y) = 0$, i.e. we get an indeterminate situation, and not an equivalence as would give Formula (9). □

It is worthwhile noticing that we put here to doubt, either the validation, *or*, the invalidation of a global outranking situation, and this in precisely two exclusive (bipolar: – *true* and *false*) situations:

1. A significant majority of criteria in favour of validating a global outranking situation is confronted with a serious counter-performance on some discordant criterion, i.e. $\exists i \in F : r(x \lll_i y) = 1$;
2. A significant majority of criteria in disfavour of validating a global outranking situation is confronted with an excellent performance on some concordant criterion, i.e. $\exists j \in F : r(y \lll_j x) = 1$.

3.3 The codual of the bipolar outranking relation

Let us now show that the codual of the outranking relation with the bipolarly extended veto principle is indeed equal to its asymmetric part, which is on turn equal to the strict bipolar outranking relation.

Let $\widetilde{\succ}_a \equiv (\widetilde{\succcurlyeq} \wedge \widetilde{\not\preccurlyeq})$ denote the asymmetric part of a bipolar outranking relation $\widetilde{\succcurlyeq}$, and $\widetilde{\succ}_{cd} \equiv \widetilde{\not\succcurlyeq}^{-1}$ its codual relation, i.e. the converse of its negation. If we define the strict bipolar outranking relation, denoted $\widetilde{\succ}_s$, as follows:

$$r(x \widetilde{\succ}_s y) \;=\; max\big(\, min(r(x > y), r(x \not\lll y)), r(x \ggg y) \,\big) \tag{11}$$

we obtain the following identities:

Proposition 3.2

$$r(x \,\widetilde{\succ}_a\, y) \;=\; r(x \,\widetilde{\succ}_{cd}\, y) \;=\; r(x \,\widetilde{\succ}_s\, y), \quad \forall (x,y) \in A^2 \tag{12}$$

Proof.

$$\begin{aligned}
r(x \,\widetilde{\succ}_a\, y) &= \min\big(r(x \,\widetilde{\succcurlyeq}\, y), r(y \;\widetilde{\not\succcurlyeq}\; x), \big) \\
&= \min\big(r(x \,\widetilde{\succcurlyeq}\, y), r(x \,\widetilde{\succ}_{cd}\, y), \big) \\
&= r(x \,\widetilde{\succ}_{cd}\, y). \\
r(x \,\widetilde{\succ}_{cd}\, y) &= r(y \;\widetilde{\not\succcurlyeq}\; x) \\
&= -r(y \,\widetilde{\succcurlyeq}\, x) \\
&= -\big[\max\big(min(r(y \geqslant x), r(y \not\lll x)), r(y \ggg x)\big)\big] \\
&= \min\big(\max(r(y \not\geqslant x), r(y \lll x)), r(y \not\ggg x)\big) \\
&= \max\big(\min(r(x > y), r(x \not\lll y)), r(x \ggg y)\big) \\
&= r(x \,\widetilde{\succ}_s\, y)
\end{aligned}$$

□

4 Conclusion

In this research note we have introduced a new bipolar veto principle which allows us to construct an extended bipolar outranking relation which guarantees the formal identity of the corresponding strict outranking relation with its asymmetric part and its associated codual relation. Contrary to the classic outranking relation, where an incomparability situation captures the difficulty to compensate excellent performances with serious counter-performances, here we rely on the neutral value of the bipolar characteristic calculus for expressing our doubts concerning the effective compensation of such contrasted performances.

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