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On Defining and Computing Fuzzy Kernels on $\mathcal{L}$-Valued Simple Graphs

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In this paper we introduce the concept of fuzzy kernels defined on $\mathcal{L}$-valued finite simple graphs in a sense close to fuzzy preference modelling. First we recall the classic concept of kernel associated with a crisp binary relation defined on a finite set. In a second part, we introduce $\mathcal{L}$-fuzzy binary relations. In a third part, we generalize the crisp kernel concept to such $\mathcal{L}$-fuzzy binary relations and in a last part, we present an application to fuzzy choice functions on fuzzy outranking relations.

1 Introduction

In this paper we introduce the concept of fuzzy kernel defined on $\mathcal{L}$-valued binary relations in a sense close to fuzzy preference modelling (cf. [5], [4]). First we introduce the concept of kernel associated to a crisp simple graph. In a second part we introduce in some detail $\mathcal{L}$-valued binary relations where $\mathcal{L}$ is a symmetric evaluation domain for a credibility calculus introduced on binary relation. In a third part we generalize the kernel concept to such $\mathcal{L}$-valued simple graphs, that is $\mathcal{L}$-valued binary relations on finite sets and in a last part, we present the application to fuzzy choice functions on fuzzy outranking relations as used in the context of multicriterian decision aid (cf. [7]).

Let $G(A, R)$ be a simple graph with R being a crisp binary relation on a finite set A of dimension n. A subset Y_R of A is an *dominant (initial) or absorbent (terminal) kernel* of the graph G, if it verifies conjointly the following right and left interior stability and corresponding exterior stability conditions:

right interior stability:

$\forall\, a, b \in A\ (a \neq b)\!: ((a, b) \in R)) \wedge (b \in Y_R) \Rightarrow (a \notin Y_R).$

left interior stability:

$\forall\, a, b \in A\ (a \neq b)\!: ((b, a) \in R)) \wedge (b \in Y_R) \Rightarrow (a \notin Y_R).$

initial or dominant (exterior) stability:

$\forall\, a \in A : (a \notin Y_R) \Rightarrow (\exists\, b \in A : (b \in Y_R)) \wedge (b, a) \in R).$

terminal or absorbent (exterior) stability:

$\forall\, a \in A : (a \notin Y_R) \Rightarrow (\exists\, b \in A : (b \in Y_R)) \wedge (a, b) \in R).$

Terminal kernels on simple graphs were originally introduced by J. Von Neumann and O. Morgenstern [11] under the name 'game solution' in the context of game theory. J. Riguet [8] introduced the name 'noyau (kernel)' for the Von Neumann 'game solution' and B. Roy [6], [7] introduced the reversed terminal or initial kernel construction as possible dominant choice procedure in the context of the multicriteria Electre decision methods. Terminal kernels were studied by C. Berge [1], [2] in the context of the Nim game modelling. Recent results on such terminal kernels on graphs for solutions of different games have been reported by G. Schmidt and T. Ströhlein [9], [10]. Finally, L. Kitaïnik [5] has recently deeply investigated dominant kernels as candidates for fuzzy choice procedures.

2 Introducing $\mathcal{L}$-fuzzy binary relations

2.1 *Basic denotational semantics for credibility calculus on binary relations*

To be able to manipulate the credibility of a particular relation between elements of some finite sets, we need to introduce a completely ordered discrete set V of degrees of credibility given by the rational interval [0, 1]. We suppose that V furthermore supports min and max operations, implementing convenient t-norms and t-conorms. In order to be able to deal with dichotomous choice representations, as needed later on in the kernel construction, we will furthermore assume on V an unary strong negation operation $\neg$ defined as follows : $\forall\, v \in V, \neg v = 1 - v$. As our kernel concept will rely on an implicative construction, we introduce on V an implication operator defined as binary operation $\rightarrow$ on V verifying following conditions: *i)* operation $\rightarrow$ is antitonic in the first and isotonic in the second argument and *ii)* $\forall\, u \in V, u \rightarrow u = 1$. Important properties follow from our conceptual choices (cf. [3] pp 154-171, [4] pp 28-31): *i)* operation $\neg$ in V is an antitonic bijection on V; *ii)* the operator triplet (min, max, $\neg$) gives a De Morgan triplet on V; *iii)* 1/2 is the unique $\neg$-fixpoint in V and *iv)* $\forall\, u, v \in V, (u \leq v) \Leftrightarrow (u \rightarrow v) = 1$.

We call the complete algebraic structure $\mathcal{L} = (V, \leq, \min, \max, \neg, \rightarrow, 0, 1/2, 1)$ a symmetric evaluation domain (s.e.d.) for our credibility calculus. All credibility degrees $v \in V$ such that $v > 1/2$, are designated as $\mathcal{L}$-*true* degrees supporting the *truthfulness* of a proposition and all the degrees $w \in V$ such that $w < 1/2$, are designated as $\mathcal{L}$-*false* degrees supporting the *untruthfulness* of the proposition in question. To each truth value $v \in V$ corresponds a unique untruth value $\neg v = w$ and the evaluation domain appears as symmetric w.r.t. the median undetermined truth value. A special mention must be made to the minimal three-element symmetric evaluation domain, presenting in fact a three-valued $\mathcal{L}_3$ logic with $V = \{0, 1/2, 1\}$(cf. [3]). A second, possibly infinite case, is given by the standard evaluation domain used in the context of fuzzy preference modelling, that is the bounded unit segment of the rational line $V = \{0, 1/m, 2/m, \ldots, m/2m = 1/2, \ldots, (m-1)/m, m/m = 1\}$, where m is any finite positive integer (generally $m = 100$ so that credibility values may be seen as percents).

2.2 *Formal definition of $\mathcal{L}$-valued binary relations*

Assume that A and B are given finite sets. A crisp binary relation from A to B is a subset of the cartesian product $A \times B$. If $A = B$ we speak of a relation R on A or a finite simple graph. Let $\mathcal{L} = (V, \leq, \min, \max, \neg, \rightarrow, 0, 1/2, 1)$ be a s.e.d for describing the credibility of such binary relations. To each ordered pair $(a, b) \in A \times B$, we associate an element from V. An $\mathcal{L}$-*valued binary relation* (short $\mathcal{L}$-vbr) R between finite sets A and B, is thus a function $R : A{\times}B \rightarrow V$ where $\forall\, (a, b) \in A \times B, R(a, b) = v \in V$. This function R is understood as a *degree of credibility* to which the relation R between elements $a \in A$ and $b \in B$ is assumed to hold. In this sense, $R(a, b)$ appears to be an $\mathcal{L}$-valued «*truth*» or «*untruth*» value for the proposition '$(a, b) \in R$'. If the underlying credibility values are restricted to $\{0, 1\}$, we obtain the standard denotational case of the classic boolean bivalent logic, where corresponding $\mathcal{L}$-vbr's represent standard crisp relations and where degrees 0 and 1 may be assimilated respectively to the boolean values '*true*' and '*false*'. In the sequel, we shall note $\mathcal{B} = (\{0, 1\}, \min, \max, \neg, \rightarrow, 0, 1)$ this boolean evaluation domain and our $\mathcal{L}$-valued binary relations appear as natural multi-valued generalization of classical $\mathcal{B}$-valued or 'crisp' binary relations. $\mathcal{L}$-vbr's evaluated in $\mathcal{B}$ will be called in the sequel $\mathcal{B}$-valued binary or crisp relations. We may define the *negation* $\neg R$, the *reverse* R^{-1}, $\mathcal{L}$-valued *set-union* $R \cup S$, $\mathcal{L}$-valued *set-intersection* $R \cap S$ and the implication operator on $\mathcal{L}$-vbr's R as term by term extensions of the respective $\mathcal{L}$-operators. Let us denote U the universal $\mathcal{L}$-vbr and 0 the null $\mathcal{L}$-vbr that is: $\forall\, (a, b) \in A{\times}B$, $U(a, b) = 1$ respectively $0(a, b) = 0$. We shall note $M = \neg M$ the unique fixpoint of the negation, that is the all median-valued relation M where $\forall\, (a, b) \in A{\times}B, M(a, b) = 1/2$. With respect to inclusion on crisp relations, we may define a corresponding crisp implicative ordering '$\leq$' on two $\mathcal{L}$-vbr's R and S defined on some finite product set $A{\times}B$ in the following way: $R \leq S \equiv \forall(a, b) \in A{\times}B, R(a, b) \leq S(a, b)$. Let $R^{\mathcal{L}}$ be the set of all such $\mathcal{L}$-vbr's. The ordering $(R^{\mathcal{L}}, \leq)$ is then a complete partial order with bottom element O and top element U verifying the following properties: *i)* $R \leq S \Leftrightarrow R \rightarrow S = U$; *ii)* In the $\mathcal{B}$-valued case, we recover the standard crisp inclusion order on binary relations.

Let $R : A{\times}B \rightarrow V$ and $S : B{\times}C \rightarrow V$ be two $\mathcal{L}$-vbr's defined on finite sets A, B and C. The *$\mathcal{L}$-valued relational product* or composition of R and S is an $\mathcal{L}$-vbr $R \bullet S$ defined as a function $(R \bullet S) : A{\times}C \rightarrow V$ with analogy to the standard composition of crisp relations using max and min operators on $\mathcal{L}$ respectively as valued disjunction and conjunction:

$$\forall(a, b) \in A{\times}C : (R \bullet S)(a, c) = \max_{b \in B} [\min (R(a, b), S(b, c))].$$

2.3 Sharpness of $\mathcal{L}$-valued binary relations and associated cut relations

We shall see in the sequel that the trivial constant median-valued $\mathcal{L}$-vbr M, in fact an $\mathcal{L}$-vbr being undetermined from the credibility point of view of the modelled relations, plays an important and central role in the kernel construction below. In the same sense is it interesting for our further purpose, to distinguish logically determined $\mathcal{L}$-vbr's, that is relations that do not contain any undetermined median value. We shall call *$\mathcal{L}$-determined* an $\mathcal{L}$-vbr R defined between given finite sets A and B if $\forall(a, b) \in A{\times}B, R(a, b) \neq 1/2$. It is worthwhile noticing that $\mathcal{B}$-valued relations are necessarily $\mathcal{L}$-determined. We shall furthermore introduce, in order to judge the more or less credibility of a kernel characterization, a reflexive «*sharpness*» relation '$\nearrow$' on $\mathcal{L}$-vbr's defined in the following way. Let $R : A{\times}B \rightarrow V$ and $S : A{\times}B \rightarrow V$ be two $\mathcal{L}$-vbr's defined on finite sets A, B. We say that R is «*sharper*» than S, noted $S \nearrow R \equiv \forall\ (a, b) \in A{\times}B$: either $(R(a, b) \leq S(a, b) \leq 1/2)$ or $(1/2 \leq S(a, b) \leq R(a, b))$.

This sharpness relation '$\nearrow$' on the set $R^{\mathcal{L}}$ of all $\mathcal{L}$-vbr's defined between any finite sets A and B of respective dimensions n and p, gives a complete partial order $(R^{\mathcal{L}}, \nearrow)$ with the constant median-valued relation M as unique minimum element and $R^{\mathcal{B}}$, the 2^{np} possible $\mathcal{B}$-valued crisp relations between sets A and B, as the set of maximal (sharpest) elements. This property gives hint to the fact that the maximal elements in $(R^{\mathcal{L}}, \nearrow)$ are linked to standard defuzzification techniques as $\mathcal{B}$-valued α-cut relations. But this traditional α-cuts, except from the median α-cut on $\mathcal{L}$-determined relations, are not semantically compatible with the symmetric organization of the evaluation domain. In order to respect our basic denotational semantics, we will particularize the a-cut polarization as follows: Let $R : A{\times}B \rightarrow V$ be any $\mathcal{L}$-vbr.$\forall\ \beta \in [1/2, 1[$,we note $R_{>\beta}$ the $\mathcal{L}_3$-valued *β-cut relation* constructed from R as follows: $\forall\ (a, b)\ A{\times}B$, $R_{>\beta}(a, b) = 1 \Leftrightarrow R(a, b) > \beta$, $R_{>\beta}(a, b) = 0 \Leftrightarrow R(a, b) < \neg\beta$ and $R_{>\beta}(a, b) = 1/2 \Leftrightarrow \neg\beta \leq R(a, b) \leq \beta$. We shall be concerned mainly by the *median-cut relation* noted $R_{>\beta=1/2}$ as this $R_{>\beta=1/2}$ is the maximal element associated to R in the sharpness relation $(R^{\mathcal{L}_3}, \nearrow)$. In this sense, the median β-cut on $\mathcal{L}$-vbr's acts as a natural *truth* or *untruth polarization functor* $\pi^{\beta}{}_{1/2}$ from the category formed by the sets of all $\mathcal{L}$-vbr's defined between any finite sets to the category formed by the sets $R^{\mathcal{L}_3}$ of all $\mathcal{L}_3$-valued relations defined between these same finite sets. We may notice that the restriction to $\mathcal{L}$-determined $\mathcal{L}$-vbr's appears as an equalizer for median α- and β-cut polarization.

Finally, we shall call two $\mathcal{L}$-vbr's R and S, defined on product set $A{\times}B$, to be of «*same shape*» , noted $R \equiv_{1/2} S$, if R and S are associated with the same median β-cut relation: $(R \equiv_{1/2} S) \equiv (R_{>\beta=1/2} = S_{>\beta=1/2})$.

3 Defining kernels on $\mathcal{L}$-valued simple graphs

3.1 $\mathcal{L}$-valued formulations of interior and exterior stability conditions

Let $\{\kappa_R\}$ be a singleton set. We assume Y_R to be an $\mathcal{L}$-vbr defined on $A \times \{\kappa_R\}$, that is a function $Y_R : A \times \{\kappa_R\} \rightarrow V$, where each $Y_R(a, \kappa_R)$ for $\forall\ a \in A$, is supposed to indicate the degree of credibility of the proposition that the 'element a is included in the kernel κ_R'. As κ_R is a constant, we will simplify our notation by dropping the second argument and in the sequel $Y_R(a)$, $\forall a \in A$, is to be seen as an $\mathcal{L}$-valued characteristic vector for the kernel membership function defined on a given $\mathcal{L}$-vbr R.

As degrees of credibility of the propositions that 'a is a *right (respectively left) interior stable* element of A' we choose a value $Y_R(a)$ verifying the following conditions:

$$\max_{b \in A,\,(a \neq b)} [\min(R(a, b), Y_R(b))] \rightarrow \neg Y_R(a) = 1.$$

$$\max_{b \in A,\,(a \neq b)} [\min(R^{-1}(a, b), Y_R(b))] \rightarrow \neg Y_R(a) = 1.$$

where $\neg Y_R$ represents the $\mathcal{L}$-negation of Y_R. And similarly, as degrees of credibility $Y_R(a)$ of the propositions that 'a is an *initial (respectively terminal) stable* element of A' we choose a value $Y_R(a)$ verifying the following respective condition:

$$\max_{b \in A,\,(a \neq b)} \left[\min(R(a, b), Y_R(b)) \right] \leftarrow \neg Y_R(a) = 1.$$

$$\max_{b \in A,\,(a \neq b)} \left[\min(R^{-1}(a, b), Y_R(b)) \right] \leftarrow \neg Y_R(a) = 1.$$

In general, for any given $\mathcal{L}$-vbr R defined on a finite set A, these Y_R values are not unique, and we shall investigate in the sequel the admissible solution sets for convenient conjunctions of these different stability conditions. But first, it is worthwhile noticing that these conditions may be naturally expressed in a synthetical way with the help of relational $\mathcal{L}$-valued products and inequations.

i) Y_R is *right interior stable* $\Leftrightarrow$ $R' \bullet Y_R \leq \overline{Y}_R$;

ii) Y_R is *left interior stable* $\Leftrightarrow$ $R'^{-1} \bullet Y_R \leq \overline{Y}_R$;

iii) Y_R is *absorbent stable* $\Leftrightarrow$ $R' \bullet Y_R \geq \overline{Y}_R$;

iv) Y_R is *dominant stable* $\Leftrightarrow$ $R'^{-1} \bullet Y_R \geq \overline{Y}_R$;

where $\overline{Y}_R$ represents the $\mathcal{L}$-negation of Y_R and R' is constructed from R in the following way: $\forall\, a, b \in A$ *with* $(a \neq b)$, $R'(a, b) = R(a, b)$ and $\forall\, a \in A$, $R'(a, a) \in$ [0, 1/2].

In the $\mathcal{B}$-valued case, there is only one possible way to make R irreflexive as is required by the interior stability and this gives a sharp irreflexive relation. In our $\mathcal{L}$-valued case however, in order to assure functional completeness of our kernel constructions, an $\mathcal{L}$-valued antireflexivity transform as defined above appears more convenient (cf. subsection 3.3 below).

3.2 *Definition of $\mathcal{L}$-valued dominant or absorbent kernels*

On the basis of the above stated stability conditions, we may now generalize the concept of dominant or absorbent kernel as follows:

i) Y_R^{ra} is a *right absorbent (terminal) $\mathcal{L}$-valued* kernel if
$Y_R^{ra} = max(\nearrow)\{Y_R : (R' \bullet Y_R \leq \overline{Y}_R) \wedge (R' \bullet Y_R \geq \overline{Y}_R)\}$;

ii) Y_R^{rd} is a *right dominant (initial) $\mathcal{L}$-valued* kernel if
$Y_R^{rd} = max(\nearrow)\{Y_R : (R' \bullet Y_R \leq \overline{Y}_R) \wedge (R'^{-1} \bullet Y_R \geq \overline{Y}_R)\}$;

iii) Y_R^{la} is a *left absorbent (terminal) $\mathcal{L}$-valued* kernel if
$Y_R^{la} = max(\nearrow)\{Y_R : (R'^{-1} \bullet Y_R \leq \overline{Y}_R) \wedge (R' \bullet Y_R \geq \overline{Y}_R)\}$;

iv) Y_R^{ld} is a *left dominant (initial) $\mathcal{L}$-valued* kernel if
$Y_R^{ld} = max(\nearrow)\{Y_R : (R'^{-1} \bullet Y_R \leq \overline{Y}_R) \wedge (R'^{-1} \bullet Y_R \geq \overline{Y}_R)\}$.

We denote K_R^k with $k = \{ra, rd, la, ld\}$ the different solution sets for the corresponding $\mathcal{L}$-valued relational inequality systems, where R' represents the $\mathcal{L}$-anti-reflexive transform of R. For a given $\mathcal{L}$-vbr R, we shall call the set $K_R^d = \{Y_R : Y_R = max(\nearrow)(K_R^{rd} \cup K_R^{ld})\}$ its *dominant kernels* and the set $K_R^a = \{Y_R : Y_R = max(\nearrow)(K_R^{ra} \cup K_R^{la})\}$ its *absorbent kernels*. One may see our kernel definitions as residual constructions, in the sense that we consider as dominant or absorbent kernel candidates, only the maximal sharpest admissible kernel solutions.

3.3 *General characterization*

Due to space limitations, we will present in this subsection only the most important theoretical results concerning general characterization of $\mathcal{L}$-valued kernel solutions sets as defined above. First, a functional completeness of our four kernel constructions is achieved through the property that the trivial median valued kernel evaluation is an admissible solution for any given $\mathcal{L}$-vbr. Going a step further, we may notice that for a given kernel solution, all possibly less sharper kernel evaluations are again admissible kernel solutions, a consequence of the floating anti-reflexive diagonal terms in R'. Formally, let $Y_R^k \in \Psi_R^k$ where $k = \{ra, rd, la, ld\}$ be the respective admissible $\mathcal{L}$-valued kernel solutions sets. Let Y'^k_R be any $\mathcal{L}$-valued kernel evaluation such that $Y'^k_R \nearrow Y_R^k$, then $Y'^k_R \in \Psi_R^k$. It thus appears, that the admissible kernel solution sets Ψ_R^k are organized as lower closed chains in the sense of increasing sharpness from the always present trivial all median-valued solution to the specifically observed maximal sharpest solutions.

For any s.e.d. $\mathcal{L}$, let $\mathbf{Grph}^{\mathcal{L}}$ be the category of all $\mathcal{L}$-valued simple finite graphs, in fact the subcategory of $\mathbf{Rel}^{\mathcal{L}}$ concerned by homogeneous $\mathcal{L}$-vbr's on finite sets. Let π^β, with $\beta \in [1/2,1[$, be the β-cut functor from $\mathbf{Grph}^{\mathcal{L}}$ to $\mathbf{Grph}^{\mathcal{L}3}$, the category of all $\mathcal{L}_3$-valued simple finite graphs. Then then median β-cut functor $\pi^\beta_{1/2}$ is the only natural (in a categorical sense) β-cut functor from $\mathbf{Grph}^{\mathcal{L}}$ to $\mathbf{Grph}^{\mathcal{L}3}$. As a consequence, we may either compute $\mathcal{L}$-valued kernels and apply a β-cut to the $\mathcal{L}$-valued maximal solutions, or apply a β-cut to the initial relation and compute the corresponding $\mathcal{L}_3$-valued kernels. In this sense, the kernel solution sets for $\equiv_{1/2}$-comparable $\mathcal{L}$-vbr's (of same «shape»), will give $\equiv_{1/2}$-comparable $\mathcal{L}$-valued kernels of the same median β-cut «quotient-class» as the corresponding $\mathcal{L}_3$-valued kernels on $R_{>\beta=1/2}$.

Furthermore, as a corollary of the above result, we may observe that the kernel constructions are monotone w.r.t. the sharpness ordering '$\nearrow$' on $\mathcal{L}$-vbr's introduced in subsection 2.3 . Formally, let R and S be two $\mathcal{L}$-vbr's defined on a given finite set A with $S \nearrow R$ and let K_R^k and K_S^k with $k = \{dominant, absorbent\}$ be the respective maximal kernel solutions defined on R and S. Then $\forall\, Y_S^k \in K_S^k$, $\exists\, Y_R^k \in K_R^k$ such that $Y_S^k \nearrow Y_R^k$. Sharpening thus, in the sense of augmenting $\mathcal{L}$-true credibility degrees and diminishing $\mathcal{L}$-untrue ones, the credibility degrees associated to a given $\mathcal{L}$-vbr R, does eventually sharpen in the same sense, the resulting kernel solutions Y_R^k without altering the shape of the solutions. To effectively alter the shape of the solutions, one must change $\mathcal{L}$-true values to $\mathcal{L}$-untrue or $\mathcal{L}$-undetermined ones and vice versa.

Finally, one may observe that we recover partly the commuting property of the median β-cut polarization with the kernel constructions, in the sense that β-cutting, at a given credibility level the maximal kernel solutions, gives, except form eventually newly appearing kernel solutions, the same result as constructing the maximal kernel solutions directly on the β-cut relation.

4 $\mathcal{L}$-valued dominant kernels as choice functions on general fuzzy outranking relations

In order to illustrate the above results, we propose to consider a numerical example based on the preference data from a well known car selection problem in multicriteria literature concerning the Electre I method. (cf. [4], [7]). The considered set A of alternatives contains 8 possible decision actions: $A = \{a, b, c, d, e, f, g, h\}$. The $\mathcal{L}$-vbr R defined on this set A corresponds to an outranking index, constructed as two digits decimal truth values for the outranking relation supposed to hold on the set of alternatives. The dummy reflexive part '_' of the relation is confined to take only $\mathcal{L}$-antireflexive values.

```
R = [[ _, 75, 70,  62,   0,  0,  0,  0],
     [76,  _, 90, 100,  82, 82, 82, 80],
     [70, 86,  _, 100, 100, 46, 80, 91],
     [64, 65, 94,   _,  88, 22, 94, 74],
     [33, 57, 93, 100,   _,  0, 80, 86],
     [ 0, 73, 64,  92,  76,  _, 96, 80],
     [ 0, 63, 73,  85,  82, 70,  _, 81],
     [ 0, 60, 64,  60,  77,  0,  0,  _]]
```

The associated integer evaluation domain uses credibility values expressed as two digits integer percents, that is $V = \{0, 1, \dots 50, \dots, 99, 100\}$, and for instance $R(a, b) = 75$ means here that the truth value or credibility of the proposition '*alternative a is outranking alternative b*' is supposed to be 75%. To implement effective computation of $\mathcal{L}$-valued dominant kernels we use constraint finite domains enumeration techniques as encountered in the context of constraint logic programming (cf. [13]). The so computed median β-cut dominant kernel solutions for the above $\mathcal{L}$-vbr R are the following:

```
K_{R>1/2}^d = [  [  0, 100, 0, 0, 0,   0,   0, 0],
                 [100,   0, 0, 0, 0, 100,   0, 0],
                 [100,   0, 0, 0, 0,   0, 100, 0]].
```

The internal relational structure of the corresponding $\mathcal{L}$-valued dominant kernel solution set appears as three-dimensional complete $\nearrow$-semilattice with three maximal $\mathcal{L}$-valued-dominant kernel solutions of same shape as the above β-cut ones:

```
K_R^d = [  [24, 76, 24, 24, 24, 24, 24, 24],
           [70, 30, 30, 30, 30, 30, 70, 30],
           [70, 30, 30, 30, 30, 70, 30, 30]].
```

The overall sharpest dominant kernel is selecting the action $\{b\}$ with credibility 76%, rejecting on the other hand all other actions with credibility 100 - 76 = 24%. The other two maximal dominant kernel solutions, equivalent in overall credibility, select either the couple $\{a, f\}$ or the couple $\{a, g\}$ with same credibility of 70%. Overall $\mathcal{L}$-valued computation time on a CRAY CS6412 with the CHIP system (cf. [12]) is about 18 seconds with approximately 9 seconds for generating the right respectively the left dominant maximal solutions.

5 Conclusion

We have generalized in this paper the crisp kernel concept to fuzzy kernels on $\mathcal{L}$-valued finite simple graphs. First we have introduced the concept $\mathcal{L}$ of symmetric evaluation domain, pointing out the importance of the logically undetermined median 1/2 value. In a second part, we have introduced $\mathcal{L}$-valued binary relations and an important sharpness ordering associated with these relations where the trivial all median 1/2-valued relation M appears as general bottom or 'fuzziest' relation and classic crisp relations as maximal 'sharpest' relations. To achieve an $\mathcal{L}$-compatible defuzzification, we generalized the standard α-cut technique to a trivalent $\mathcal{L}_3$ -valued β-cut one. Our corresponding $\mathcal{L}$-valued kernel definition appears then as a relational algebraic residual construction on the basis of this earlier introduced sharpness ordering, generalizing in a natural way, the classic crisp kernel definition to our $\mathcal{L}$-valued simple graphs.

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DomFLIP++

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DomFLIP++ is the knowledge engineering module of the *FLIP++ (pronounce: StarFlipPlusPlus) project. *FLIP++ is a tool for optimizing multiple criteria problems. It uses fuzzy constraints to model optimizing criteria and applies algorithms such as Tabu search or genetic algorithms to the problems. DomFLIP++ is a C++ library. It allows the definition of new optimization problems. It helps a domain engineer to design the structure of a new problem. However, there is a domain independent interface to other *FLIP++ modules such as OptiFLIP++, DynaFLIP++, and InterFLIP++. After each iteration in the optimization process, the considered instantiations of the problem are evaluated. Each evaluation produces a list of violated constraints. For each constraint, a list of repair steps is defined that can be used to increase the score of this constraint in further iterations of the optimization. A domain can be fine-tuned through modifications of constraints, through editing their repair lists, and through change in the optimizing parameters. A well-tuned domain can be successfully applied for optimization. Object-oriented design and implementation makes this module easy to modify and to reuse. Definition of new domains, system extensions with new optimizing algorithms, and definition of specific domain-dependent repair steps can be done efficiently. DomFLIP++ is tested on real-world example, namely scheduling the steel plant LD3 in Linz, Austria.

1 Introduction

DomFLIP++ is part of the *FLIP++ library[1,2] for real-world decision making. The *FLIP++ library allows optimizing under vague constraints of different importance using uncertain data, where compromises between antagonistic criteria can be modeled. Typical application areas include scheduling, design, configuration, planning, and classification.

1.1 *FLIP++ as part of a scheduling project*

*FLIP++ is composed of the following layered sub-libraries:

- FLIP++: the basic fuzzy logic inference processor library.
- ConFLIP++: the static fuzzy constraint library.
- DynaFLIP++: the dynamic fuzzy constraint generation library.
- DomFLIP++: the domain knowledge representation library.