On Clustering the criteria in an MCDA

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Motivation

Example (Ronda decision problem)
A family, staying during their holidays in Ronda (Andalusia), is planning the next day’s activity.

The set of alternatives

<table>
<thead>
<tr>
<th>Identifier</th>
<th>Name</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>ant</td>
<td>Antequerra</td>
<td>An afternoon excursion to Antequerra and surroundings.</td>
</tr>
<tr>
<td>ard</td>
<td>Ardales</td>
<td>An afternoon excursion to Ardales and El Chorro.</td>
</tr>
<tr>
<td>be</td>
<td>beach</td>
<td>Sun, fun and more.</td>
</tr>
<tr>
<td>crd</td>
<td>Cordoba</td>
<td>A whole day visit by car to Cordoba.</td>
</tr>
<tr>
<td>dn</td>
<td>fa niente</td>
<td>Doing nothing.</td>
</tr>
<tr>
<td>lw</td>
<td>long walk</td>
<td>A whole day hiking.</td>
</tr>
<tr>
<td>mal</td>
<td>Malaga</td>
<td>A whole day visit by car to Malaga.</td>
</tr>
<tr>
<td>sev</td>
<td>Sevilla</td>
<td>A whole day visit by car to Sevilla.</td>
</tr>
<tr>
<td>sw</td>
<td>short walk</td>
<td>Less than a half day hiking.</td>
</tr>
</tbody>
</table>

Example (Ronda example – continued)
The family members agree to measure their preferences with respect to the following set of criteria:

The family of criteria

<table>
<thead>
<tr>
<th>Identifier</th>
<th>Name</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>cult</td>
<td>Cultural Interest</td>
<td>Andalusian heritage.</td>
</tr>
<tr>
<td>dis</td>
<td>Distance</td>
<td>Minutes by car to go to and come back from the activity.</td>
</tr>
<tr>
<td>food</td>
<td>Food Quality</td>
<td>Quality of the expected food opportunities.</td>
</tr>
<tr>
<td>sun</td>
<td>Sun, Fun, &amp; more</td>
<td>No comment.</td>
</tr>
<tr>
<td>phy</td>
<td>Physical Investment</td>
<td>Contribution to physical health care.</td>
</tr>
<tr>
<td>rel</td>
<td>Relaxation</td>
<td>Anti-stress support.</td>
</tr>
<tr>
<td>tour</td>
<td>Tourist Attraction</td>
<td>How many stars in the guide?</td>
</tr>
</tbody>
</table>
Example (Ronda example – continued)

The common evaluation of the performances of the nine alternatives on all the criteria results in the performance table shown here:

<table>
<thead>
<tr>
<th>Criteria</th>
<th>ant</th>
<th>ard</th>
<th>be</th>
<th>crd</th>
<th>dn</th>
<th>lw</th>
<th>mal</th>
<th>sev</th>
<th>sw</th>
</tr>
</thead>
<tbody>
<tr>
<td>cult</td>
<td>7</td>
<td>3</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>dis</td>
<td>120</td>
<td>100</td>
<td>90</td>
<td>360</td>
<td>0</td>
<td>90</td>
<td>240</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>phy</td>
<td>3</td>
<td>7</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>10</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>rel</td>
<td>1</td>
<td>5</td>
<td>8</td>
<td>3</td>
<td>10</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>food</td>
<td>8</td>
<td>10</td>
<td>4</td>
<td>8</td>
<td>10</td>
<td>1</td>
<td>8</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>sun</td>
<td>0</td>
<td>3</td>
<td>10</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>tour</td>
<td>5</td>
<td>7</td>
<td>3</td>
<td>10</td>
<td>0</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>5</td>
</tr>
</tbody>
</table>

All performances on the qualitative criteria are marked on a same ordinal scale going from 0 (lowest) to 10 (highest). On the quantitative \textit{Distance} criterion (to be minimized), the required travel time to go to and return from the activity is marked in minutes.

In order to model only effective preferences, an \textit{indifference threshold} of 1 point and a \textit{preference threshold} of 2 points is put on the qualitative performance measures.

On the \textit{Distance} criterion, an indifference threshold of 20 min., and a preference threshold of 45 min. is considered.

Furthermore, a difference of more than two hours (\(\geq 121\) min.) to attend the activity’s place is considered raising a veto.

Example (Ronda example – continued)

How do the criteria express their preferential view point on the set of alternatives?

For instance the \textit{Tourist Attraction} criterion appears to be in its preferential judgments somehow positively correlated with both the \textit{Cultural Interest} and the \textit{Food Quality} criteria.

It is also apparent that the \textit{Distance} criterion is somehow negatively correlated to these latter criteria.

How can we explore and illustrate these intuitions?
Notations

- We consider a finite set $A$ of $n$ alternatives and denote by $x$ and $y$ any two alternatives.
- We consider also a set $F$ of outranking criteria denoted by variables $i$ or $j$.
- with $k = 0,1,...$ discrimination thresholds.
- The performance of an alternative $x$ on criterion $i$ is denoted by $x_i$.

Example: Rubis discrimination thresholds

The four discrimination thresholds we may observe, for instance, on each criterion $i$ in the Rubis choice method are:

- "weak preference" $wp_i$ ($0 < wp_i$),
- "preference" $p_i$ ($wp_i \leq p_i$),
- "weak veto" $wv_i$ ($p_i < wv_i$), and
- "veto" $v_i$ ($wv_i \leq v_i$).

Homogeneous semiorders

In general, let us consider on each criterion $i$, supporting a set of discrimination thresholds $p^k_i$ ($r = 1,..,k$) such that $0 < p^1_i \leq ... \leq p^k_i$, the Kendall vector (Degenne 1972) gathering the classification of all possible differences $(x_i - y_i)$ into one of the following $2k + 1$ categories:

$$(x_i - y_i) \in \begin{cases} 
(=) & \text{if } -p^1_i < (x_i - y_i) \leq -p^r_i, \text{ for } r = 1,...k-1 \\
(<r) & \text{if } -p^r_i < (x_i - y_i) < p^{r+1}_i, \text{ for } r = 1,...k-1 \\
(>) & \text{if } p^k_i \leq (x_i - y_i) \\
(>k) & \text{if } (x_i - y_i) \leq -p^k_i 
\end{cases} \quad (1)$$

Example: Rubis discrimination thresholds

Each performance difference $(x_i - y_i)$ may thus be classified into one and only one of the following nine categories:

$$(\gg) \ "\text{veto against } x \leq y" \quad \Leftrightarrow \quad v_i \leq (x_i - y_i)$$
$$(\geq) \ "\text{weak veto against } x \leq y" \quad \Leftrightarrow \quad wv_i \leq (x_i - y_i) < v_i$$
$$(>) \ "x \ better than y" \quad \Leftrightarrow \quad p_i \leq (x_i - y_i) < wv_i$$
$$(\geq) \ "x \ better than or equal y" \quad \Leftrightarrow \quad wp_i \leq (x_i - y_i) < p_i$$
$$(=) \ "x \ indifferent to y" \quad \Leftrightarrow \quad -wp_i < (x_i - y_i) < wp_i$$
$$(\leq) \ "x \ worse than or indifferent to y" \quad \Leftrightarrow \quad -p_i < (x_i - y_i) \leq -wp_i$$
$$(<) \ "x \ worse than y" \quad \Leftrightarrow \quad -wp_i < (x_i - y_i) < -p_i$$
$$(\ll) \ "\text{weak veto against } x \geq y" \quad \Leftrightarrow \quad -v_i < (x_i - y_i) \leq -wp_i$$
$$(\lll) \ "\text{veto against } x \geq y" \quad \Leftrightarrow \quad (x_i - y_i) \leq -v_i$$
A bipolar-valued ordinal correlation index

- Considering criteria $i$ and $j$, we say that $x$ and $y$ are *concordantly* (resp. *discordantly*) compared if $(x_i - y_i)$ and $(x_j - y_j)$ are classified into the same category (resp. different categories) on both criteria.
- There are $n(n-1)$ distinct ordered pairs of performances and each pair $(x, y)$ is thus either concordantly or discordantly classified.
- Denoting by $S_{ij}$ the number $c_{ij}$ of concordantly classified minus the number $d_{ij}$ of discordantly classified ordered pairs, the *ordinal criteria correlation index* $\tilde{T}$ is defined on $F \times F$ as

$$\tilde{T}(i,j) = \frac{c_{ij} - d_{ij}}{c_{ij} + d_{ij}} = \frac{S_{ij}}{n(n-1)}.$$

A bipolar-valued correlation index

**Property (1)**

The ordinal criteria correlation index $\tilde{T}$ is symmetrically valued in the rational bipolar credibility domain $[-1, 1]$.

**Proof.**

- If $d_{ij} = 0$ (resp. $c_{ij} = 0$), $\tilde{T}(i,j) = 1.0$ (resp. $-1.0$).
- If $\tilde{T}(i,j) > 0$ (resp. $< 0$) both criteria are more *similar than dissimilar* (resp. *dissimilar than similar*) in their preferential judgments. When $\tilde{T}(i,j) = 0$, no conclusion can be drawn.
- A performance difference $(x_i - y_i)$ is classified in one and only one category.
- The category of $(x_i - y_i)$ corresponds bijectively to a unique symmetric category of $(y_i - x_i)$.

**Example (The Ronda example – continued)**

Let $\Delta_\epsilon$ denote the smallest observable difference in a given performance table. If $i$ and $j$ admit single preference thresholds $p_i^1, p_j^1 \leq \Delta_\epsilon$ and we don’t observe ties in the performance table, then $\tilde{T}(i,j)$ is identical with the classical rank correlation index $\tau$ of Kendall (1938).

**Proof.**

- Either $(>)_1$: $(x_i - y_i) \geq \Delta_\epsilon$, or $(<)_1$: $(x_i - y_i) \leq \Delta_\epsilon$.
- Let $p_{ij}$ be the number of pairs $(x, y)$ in $A \times A$ such that conjointly $(x_i - y_i) \geq \Delta_\epsilon$ and $(x_j - y_j) \geq \Delta_\epsilon$:

$$\tilde{T}(i,j) = (2 \times \frac{2p_{ij}}{n(n-1)}) - 1, \quad \forall (i,j) \in F \times F,$$

i.e. Kendall’s original $\tau$ definition.
A bipolar-valued similarity digraph

- \( \tilde{T} \) represents a bipolar-valued characteristic denotation of the propositional statement "criteria i and j express similar preferential statements on A".
- We consider indeed this statement to be more or less validated if both criteria are concordant on a majority of pairwise comparisons and discordant on a minority ones.
- \( \tilde{T} \) is characterising a bipolar-valued similarity graph, we denote by \( \tilde{S}(F, \tilde{T}) \) or \( \tilde{S} \) for short.
- Following from the logical denotation of the bipolar valuation, we say that there is an arc between i and j if \( \tilde{T}(i,j) > 0 \).

Motivation

Example (The Ronda example – continued)

In general, we may associate a crisp graph \( S(F, T) \) with \( \tilde{S} \), where \( T = \{(i,j)|\tilde{T}(i,j) > 0\} \).
- All properties of \( S \) are canonically transferred to \( \tilde{S} \).
- For instance, \( S \) is a symmetric digraph, so is \( \tilde{S} \).
- Similarly, a clique \( C \) in \( \tilde{S} \) is a subset of criteria such that for all i and j in \( C \), we have \( \tilde{T}(i,j) \geq 0 \).

Example (The Ronda example: Principal Component Analysis)

-0.4 -0.2 0.0 0.2 0.4
-0.3 -0.1 0.0 0.1 0.2
axis 1: 46.1 %
axis 2: 22.6 %
factors 1 and 2
factors 2 and 3
factors 1 and 3
total inertia: 68.7 %
total inertia: 39.7 %
total inertia: 63.2 %
Example (The Ronda example: Principal Component Analysis)

Maximal bipolar-valued cliques

What we are looking for are maximal cliques, i.e. subsets $C$ of criteria which verify both the following properties:

- **Internal stability:** all criteria in $C$ are similar, i.e. the subgraph $(C, \tilde{T}|_C)$ is a clique;
- **External stability:** if a criteria $i$ is not in $C$, there must exist a criteria $j$ in $C$ such that $\tilde{T}(i,j) < 0$ and $\tilde{T}(j,i) < 0$.

**Definition (Stabilities' credibility)**

For any $C \subseteq F$, we denote by $\Delta^{\text{int}}(C)$ (resp. $\Delta^{\text{ext}}(C)$) its credibility of being internally (resp. externally) stable:

$$\Delta^{\text{int}}(C) = \begin{cases} 1.0 & \text{if } |C| = 1, \\ \min_{i \in C} \min_{j \neq i \in C} (\tilde{T}(i,j)) & \text{otherwise.} \end{cases}$$

$$\Delta^{\text{ext}}(C) = \begin{cases} 1.0 & \text{if } C = F, \\ \min_{i \in F} \max_{j \in C} ( - \tilde{T}(i,j)) & \text{otherwise.} \end{cases}$$

**Property (3)**

A subset $C$ of criteria is a maximal clique of the similarity graph $\tilde{S} \equiv (F, \tilde{T})$ if and only if both $\Delta^{\text{int}}(C) \geq 0$ and $\Delta^{\text{ext}}(C) > 0$. 

**Proof.**

- $\Delta^{\text{int}}(C) \geq 0$ implies that $(C, \tilde{T}|_C)$ is a clique
- $\Delta^{\text{ext}}(C) > 0$ implies that, for any criterion $i$ not in $C$, there exists at least one criterion $j$ in $C$ such that $\tilde{T}(i,j) < 0$. 

□
Example (The Ronda example – continued)

Clustering the criteria

<table>
<thead>
<tr>
<th>Maximal cliques</th>
<th>Credibility level (in %)</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>external</td>
</tr>
<tr>
<td>{dis}</td>
<td>80.56</td>
<td>+0.083</td>
</tr>
<tr>
<td>{rel}</td>
<td>58.33</td>
<td>+0.17</td>
</tr>
<tr>
<td>{phy,tour}</td>
<td>58.33</td>
<td>+0.17</td>
</tr>
<tr>
<td>{tour,cult}</td>
<td>58.33</td>
<td>+0.17</td>
</tr>
<tr>
<td>{sun,tour}</td>
<td>50.00</td>
<td>+0.28</td>
</tr>
<tr>
<td>{cult,food}</td>
<td>50.00</td>
<td>+0.17</td>
</tr>
</tbody>
</table>

Concluding Remarks

In this communication we have presented:
- a generalisation of Kendall’s rank correlation $\tau$ measure to homogenous semiorders;
- a bipolar ordinal criteria correlation index;
- a graphical illustration of oppositions and agreements between criteria with the help of a PCA;
- a bipolar-valued criteria similarity digraph;
- a bipolar-valued clustering of the criteria.

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