The Electre like outranking approach to MCDA

II: Recent advances

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How to specify the criteria significances?

- The numerical criteria significances play a crucial role in the construction of the bipolarly-valued outranking digraph.
- Two different approaches are mainly proposed for specifying the criteria significances:
  a. either, *directly* by knowledge or assessment,
     - Roy & Bouyssou 93;
     - Roy & Mousseau 96,
  b. or, *indirectly* via some a priori partial knowledge of the resulting global outranking relation:
     - Mousseau & Słowinski 98;
     - Meyer, Marichal & Bisdorff 08.

Indirect estimation of criteria significances (2)

Here, we focus on the indirect approach.

Similar disaggregation-aggregation or ordinal regression methods have been proposed in MAUT and MAVT contexts:

- Jacquet-Lagrèze & Siskos 82;
- Mousseau, Figueira, Dias, Gomes da Silva & Clímaco 03;
- Greco, Mousseau & Słowinski 08;
- Grabisch, Kojadinovic & Meyer 08.

In our Electre-like outranking approach, we will use, as a priori knowledge, the robustness of the CONDORCET outranking graph, i.e. the robustness of the significant majority that a decision maker acknowledges for his/her pairwise outranking comparisons (Bisdorff 04).

Notations

- Let F be a set of m performance criteria;
- Let W denote a vector of m criteria significances;
- Let $\sqsupseteq_W$ be the preorder modelled on F by the numerical $\geqslant$ relation defined on significance vector W.
- The equivalence quotient of $\sqsupseteq_W$ induces s ordered equivalence classes: $\Pi_1^W \sqsupset_W \ldots \sqsupset_W \Pi_s^W$ where $1 \leq s \leq m$; All criteria gathered in a same equivalence class have same significance.
- For $i < j$, those of Π_i^W have a higher significance than those of Π_j^W.
- If $\mathcal{W}$ represents the set of all potential significance vectors, then $\mathcal{W}_{\sqsupseteq_W} \subset \mathcal{W}$ denotes the set of all significance vectors that are preorder-compatible with $\sqsupseteq_W$.

The CONDORCET robustness denotation

Consider a bipolarly-valued outranking graph $\widetilde{G}(X, r^W(\succsim))$ using a significance vector W. For any pair (x, y) of alternatives, the CONDORCET robustness of the outranking $(x \succsim y)$, denoted $[\![(x \succsim^W y)]\!]$ is defined as follows:

1. $[\![(x \succsim^W y)]\!] = \pm 3$ if $r^W(x \succsim y) = \pm 1.0$;
2. $[\![(x \succsim^W y)]\!] = \pm 2$ if $r^W(x \succsim y) > 0.0$, resp. < 0.0, for all $\sqsupseteq_W$-compatible significance vectors;
3. $[\![(x \succsim^W y)]\!] = \pm 1$ if $r^W(x \succsim y) > 0.0$, resp. < 0.0, for some but not for all $\sqsupseteq_W$-compatible significance vectors;
4. $[\![(x \succsim^W y)]\!] = 0$ if $r^W(x \succsim y) = 0.0$.

Measuring the CONDORCET robustness

- Let $r^{\%}(x \geqslant_i y) = \big(r(x \geqslant_i y) + 1\big)/2$ be the $[0, 1]$-recoded marginal characteristic r-functions and let there be $k = 1, ..., s$ significance classes Π_k.
- Let $c_k^W(x, y)$ be the sum of "*at least as good as*" characteristics $r^{\%}(x \geqslant_i y)$ for all criteria $i \in \Pi_k^W$, and $\overline{c_k^W}(x, y)$ the sum of the negation: $1 - r^{\%}(x \geqslant_i y)$, of these characteristics.
- Furthermore, let $C_k^W(x, y) = \sum_{i=1}^{k} c_i^W(x, y)$ be the cumulative sum of "*at least as good as*" characteristics for all criteria having significance at least equal to the one associated to Π_k^W, and

  let $\overline{C_k^W}(x, y) = \sum_{i=1}^{k} \overline{c_i^W}(x, y)$ be the cumulative sum of the negation of these characteristics for all k in $\{1, \ldots, s\}$.

Measuring the CONDORCET robustness (continue)

In the absence of ± 3 denotations, the following proposition gives us a test for the presence of a $+2$ denotation:

Proposition (Bisdorff 2004, *4OR:2(4)*)

$$[\![(x \succsim^W y)]\!](x, y) = +2 \iff \begin{cases} \forall k \in 1, ..., s : C_k^W(x, y) \geqslant \overline{C_k^W}(x, y); \\ \exists k \in 1, ..., s : C_k^W(x, y) > \overline{C_k^W}(x, y). \end{cases}$$

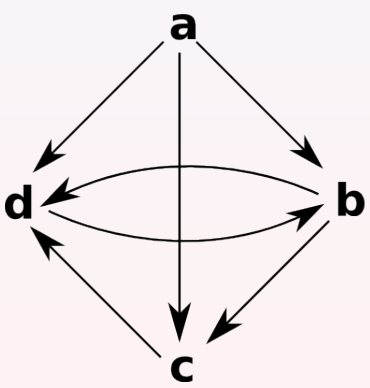
The negative -2 denotation corresponds to similar conditions with reversed inequalities.

The proof relies on the verification of first order stochastic dominance conditions.

Example of valued outranking

	1	2	3
a	10	4	8
b	5	6	4
c	7	2	3
d	5	7	2
p	1.0	1.0	1.0
W	3.0	1.5	2.0

$r(\succsim^W)$	a	b	c	d
a	-	.54	1.0	.54
b	-.54	-	.08	.54
c	-1.0	-.08	-	.54
d	-0.54	0.38	-.54	-

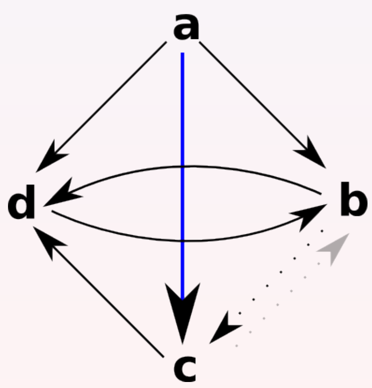


The Condorcet Outranking Digraph

Condorcet robustness

	1	2	3
p	1.0	1.0	1.0
W	**3.0**	**1.5**	**2.0**
a	10	4	8
b	5	6	4
c	7	2	3
d	5	7	2

$[\![r(\succsim^W)]\!]$	a	b	c	d
a	-	+2	**+3**	+2
b	-2	-	**+1**	+2
c	**-3**	**-1**	-	2
d	-2	+2	-2	-

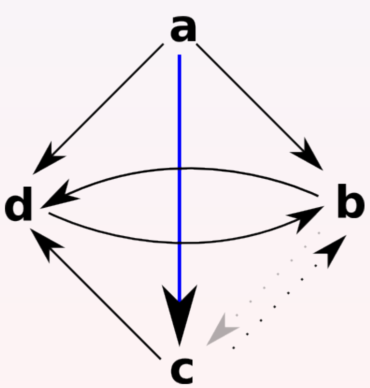


The Condorcet Outranking Digraph

Condorcet robustness

	1	2	3
p	1.0	1.0	1.0
W	**4.0**	**1.5**	**2.0**
a	10	4	8
b	5	6	4
c	7	2	3
d	5	7	2

$[\![r(\succsim^W)]\!]$	a	b	c	d
a	-	+2	+3	+2
b	-2	-	**-1**	+2
c	-3	**+1**	+3	+2
d	-2	+2	-2	-



The Condorcet Outranking Digraph

Inverse Analysis from the Condorcet robustness

In a decision aid problem we are given:

1. A set X of n decision alternatives evaluated on a set F of m performance criteria;
2. A performance table, of dimension $n \times m$, but without any precise information concerning the criteria significances.
3. Suppose we are, now, given the apparent Condorcet robustness denotation $[\![(x \succsim^W y)]\!]$, but, without actually knowning the corresponding significance vector W and, hence, the associated pairwise bipolarly-valued outranking characteristics $r(x \succsim^W y)$.

Inverse Analysis from the CONDORCET robustness

The criteria significance estimation problem

Given the marginal outranking characteristics $r(x \geqslant_i y)$ and a CONDORCET *robustness denotation $[\![(x \succsim^W y)]\!]$ for (x, y) in X^2, can we compute a preorder $\sqsupseteq$ on the criteria significances and a numerical instance W^* in $\mathcal{W}_{\sqsupseteq W}$ (the set of $\sqsupseteq$-compatible significance vectors) which satisfies $[\![(x \succsim^W y)]\!]$? In other terms:*

Knowing $r(x \geqslant_i y)$, how to choose $\sqsupseteq$ and W^* such that $[\![(x \succsim^{W^*} y)]\!] = [\![(x \succsim^W y)]\!]$?

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Estimating apparent criteria significances

The decision variables $P_{m \times M}$

- Each criterion gets an integer significance w_i in $[1, M]$, where the parameter M denotes the maximal admissible value.
- $P_{m \times M}$ is a Boolean (0, 1)-matrix, with general term $[p_{i,u}]$, that characterises row-wise the number of significance units allocated to criterion i such that: $\sum_{u=1}^{M} p_{i,u} = w_i$.
- For instance, if criterion i accepts an integer significance of 3 and if we decide that $M = 5$, then the ith row of $P_{m\times 5}$ corresponds to $(1, 1, 1, 0, 0)$.
- Each criterion must have a strictly positive significance: $\sum_{i \in F} p_{i,1} = m$,
- And the cumulative constraints require that:

$$p_{i,u} \geqslant p_{i,u+1} \quad (\forall i = 1, ..., m,\ \forall u = 1, ..., M-1).$$

The CONDORCET robustness constraint

The CONDORCET robustness test may be formulated as:

$$[\![(x \succsim^W y)]\!] = 2 \iff \begin{cases} \forall u \in 1, ..., \max w_i : C_u'^W(x,y) \geqslant \overline{C_u'^W}(x,y); \\ \exists u \in 1, ..., \max w_i : C_u'^W(x,y) > \overline{C_u'^W}(x,y); \end{cases}$$

where $C_u'^W(x,y)$ (resp. $\overline{C_u'^W}(x,y)$) is the sum of all $r^{\%}(x \geqslant_i y)$ (resp. $\overline{r}^{\%}(x \geqslant_i y) = 1 - r^{\%}(x \geqslant_i y)$) such that the significance $w_i \leq u$.

For all pairs $(x, y) \in X^2_{+2}$ we get

$$\sum_{i \in F} \left(p_{i,u} \cdot \left[r^{\%}(x \geqslant_i y) - \overline{r}^{\%}(x \geqslant_i y) \right] \right) \geqslant b_u(x,y),$$

where the $b_u(x, y)$ are Boolean $(0, 1)$ variables for each pair of alternatives and each equi-significance level u in $\{1, \ldots, M\}$, which allow us to impose at least one case of strict inequality for each $(x, y) \in X^2_{\pm 2}$: $\sum_{u=1}^{m} b_u(x,y) \geqslant 1$.

The objective function

$\min_{P_{m\times M}} O =$

$$K_1\Big(\sum_{g_i\in F}\sum_{u=1}^{M} p_{i,u}\Big) \quad \text{Minimize the sum of the weights;}$$

$$-\ K_2\Big(\sum_{u=1}^{M}\Big(\sum_{(x,y)\in A^2_{\pm 2}} b_u(x,y)\Big)\Big) \quad \text{Maximise the } \pm 2 \text{ robustness;}$$

$$+\ K_3\Big(\sum_{(x,y)\in A^2_{\pm 1}} s^{\pm 1}(x,y)\Big) + K_4\Big(\sum_{(x,y)\in A^2_0}\big(s^0_+(x,y)+s^0_-(x,y)\big)\Big)$$

Comment

- *$s^{\pm 1}$ as well as $s^0_{\pm}$ are slack variables for softening, the case given, the ± 1 and 0 robustness constraints,*
- *$K_1 \ldots K_4$ are parametric constants used for the correct hierarchical ordering of the four sub-goals.*

The mixed-integer LP model

MILP

Variables:

$p_{i,u} \in \{0,1\}$	$\forall i \in F,\ \forall u = 1,..,M$
$b_u(x,y) \in \{0,1\}$	$\forall (x,y) \in X^2_{\pm 2}, \forall u = 1,..,M$
$s^{\pm 1}(x,y) \geqslant 0$	$\forall (x,y) \in X^2_{\pm 1}$
$s^0_+(x,y) \geqslant 0\ ,\ s^0_-(x,y) \geqslant 0$	$\forall (x,y) \in X^2_0$

Parameters:

M	usually $\lceil m/2 \rceil$ or m
$K_i > 0$	$\forall i = 1 \ldots 4$

Objective function:

$$\min \quad K_1\Big(\sum_{g_i\in F}\sum_{u=1}^{M} p_{i,j}\Big) - K_2\Big(\sum_{u=1}^{M}\sum_{(x,y)\in A^2_{\pm 2}} b_u(x,y)\Big)$$
$$+K_3\Big(\textstyle\sum_{(x,y)\in A^2_{\pm 1}} s^{\pm 1}(x,y)\Big) + K_4\Big(\textstyle\sum_{(x,y)\in A^2_0}\big(s^0_+(x,y)+s^0_-(x,y)\big)\Big)$$

The mixed-integer LP model (continue)

Constraints:

$$\sum_{i\in F} p_{i,1} = m$$

$$p_{i,u} \geqslant p_{i,u+1} \qquad \forall g_i \in F,\ \forall u = 1,..,M-1$$

$$\sum_{i\in F}\Big(p_{i,u}\cdot\big[\,r^{\%}(x \geqslant_i y) - \bar r^{\%}(x \geqslant_i y)\big]\Big) \gtreqless b_u(x,y) \qquad \forall (x,y) \in X^2_{\pm 2},\ \forall u = 1,..,M$$

$$\sum_{u=1}^{M} b_u(x,y) \geqslant 1 \qquad \forall (x,y) \in X^2_{\pm 2}$$

$$\sum_{i\in F}\Big(\big(\textstyle\sum_{u=1}^{M} p_{i,u}\big)\cdot \pm\big(r^{\%}(x \geqslant_i y) - \bar r^{\%}(x \geqslant_i y)\big)\Big) \pm s^1_{\pm}(x,y) \geqslant 1 \qquad \forall (x,y) \in X^2_{\pm 1},\ \forall u = 1,..,M$$

$$\sum_{i\in F}\big(\textstyle\sum_{u=1}^{M} p_{i,u}\big)\cdot\big(r^{\%}(x \geqslant_i y) - \bar r^{\%}(x \geqslant_i y)\big) + s^0_+(x,y) - s^0_-(x,y) = 0 \qquad \forall (x,y) \in X^2_0,\ \forall u = 1,..,M$$

Result of the Inverse Analysis

	1	2	3
p	1.0	1.0	1.0
W	3.0	1.5	2.0
a	10	4	8
b	5	6	4
c	7	2	3
d	5	7	2
W^*	3.0	2.0	2.0

$r(x \succsim^W y)$	a	b	c	d
a	-	.54	1.0	.54
b	-.54	-	.08	.54
c	-1.0	-.08	-	.54
d	-0.54	0.38	-.54	-

Valued majority margins obtained with original significance vector $W = [3.0, 2.0, 1.5]$.

Cond	a	b	c	d
a	-	2	3	2
b	-2	-	-1	2
c	-3	1	3	2
d	-2	2	-2	-

$r(x \succsim^{W^*} y)$	a	b	c	d
a	-	.43	1.0	.43
b	-.43	-	.14	.43
c	-1.0	-.14	-	.43
d	-0.43	0.43	-.43	-

Valued majority margins obtained with estimated significance vector $W^* = [3, 2, 2]$.

Solving the MILP

- We solve the MILP model with Cplex associated with an AMPL front end modeler;
- On more or less real-sized random multiple criteria decision problems (20 alternatives evaluated on 13 criteria) we observe quite reasonable reasonable solving times on an 6 threaded standard application server;
- Depending on the maximal value M allowed for an individual criterion significance weight we indeed obtain:
  - average computation times of 2.5 seconds for $M = 7$,
  - up to 2 minutes for $M = 13$.

Partial preference information

Partial preference information may be easily integrated in the previous MILP model, like

1. fix or confine the a priori significance of some criterion;
2. make a criterion, or a coalition of criteria, more significant than others;
3. allocate a significant majority to a coalition of criteria.

A progressive and robust decision aid approach

1. When no information concerning the significances of the criteria is available, we solve the problem with equi-significant criteria, i.e. one single weight equivalence class.
2. Some apparent outranking situations may be aknowledged, some others not. Under this partial preference information, the most robust valued outranking relation is estimated.
3. As long as the resulting outranking digraph is too indeterminate, we may ask further partial preference information until the decision maker is satisfied with the apparent preference model.

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