On the stability of the majority-cut outranking digraph

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Abstract

In Multicriteria Decision Aid (MCDA), when working with outranking methods, the conclusion that an alternative appears being *at least as good as* another one, or not, depends on a clear setting of different parameters, especially the weights of the performance criteria. In this article, we present the concept of *stability* of the crisp majority-cut outranking digraph with respect to chosen criteria weights. We show in particular that, when a majority-cut outranking statement can be qualified as stable, it is less important to precisely quantify these criteria weights. We give an intuitive formulation, as well as simple mathematical conditions, for computing the degree of stability of outranking situations. Moreover, we propose a protocol for eliciting criteria weights that render the outranking modeling as stable as possible.

Keywords: Multiple criteria analysis, Robustness and sensitivity analysis, Stable outrankings.

1 Introduction

We consider a decision situation in which a finite set of decision alternatives is evaluated on a finite set of criteria. A decision-maker is willing to express the weights of the criteria according to the outranking paradigm, in order to assess the overall outranking relation between all pairs of alternatives. We consider indeed that an alternative x *outranks* an alternative y when a *weighted majority* of criteria validates the fact that x performs at least as good as y and there is no criterion where y seriously outperforms x **?**.

However, precisely quantifying these weights is an important issue in Multi-criteria Decision Aid (MCDA), when applying outranking methods **?**, and has a considerable impact on the decision recommendations. Being able to measure the dependency of the outranking relations with respect to the impreciseness and uncertainty related to the precise numerical values of the weights can indeed be a helpful instrument to provide robust decision recommendations **??**. In **?**, the author states that the general idea behind the robustness analysis approaches is to accept multiple model versions (or scenarios) and to try to identify a solution that is seen as being good or acceptable in (almost) every model versions. Roy, moreover, notes that a sensitivity analysis may be quite time-consuming, but that it is necessary to construct, to modify or to justify some preferences before starting a critical discussion and establishing a recommendation **?**.

A fair number of authors defined various means to perform sensitivity analyses in order to measure the impact of the impreciseness of some parameters (see for instance **?**). When it comes to analyzing the weights of the criteria, most of the time, each of the criteria is considered independently, as their weights are tested separately around an "ideal" solution. Maystre *et al* **?** suggest different techniques to take into account the interactions among the the parameters, but the resulting high combinatorial number of values to be considered makes the interpretation of the results very difficult. The authors of **?** interestingly argue that robust conclusions should be studied at the outranking relation level, and they propose solutions to enrich conclusions when the parameters suffer from impreciseness. In **?**, within the context of value functions, the authors develop an analysis software, VIP, which computes a minimal and maximal score for each alternative, under a set of linear constraints, allowing to consider interdependencies between the parameters.

In a similar manner, in response to the difficulty of tackling inaccurate or imprecise input-oriented preferential information, the authors of **?** propose to determine the set of possible outranking relations which are compatible with the given preference information, without selecting one particular model. They consequently seek robust conclusions that are in agreement with all possible models.

It has also been suggested in **?** to consider as valid only the outranking situations supported by a significant majority of criteria. However, as we will discover, the stability of an outranking relation is not directly correlated with the level of its valuation.

In this article, considering a given vector of weights on the criteria, we characterise the stability of the resulting crisp majority-cut outranking relations, namely the dependency of each validated outranking situation, with respect to the more or less precise fixation of the weights. This work extends the one in **?**, by giving a more intuitive formulation of the concept of stability and a simplified way of computing it. Furthermore, we extend the original idea by introducing a sharper characterisation of the dependencies with two additional levels of stability. We also present how an analyst may rely on this concept to simplify the determination of some criteria weights in best accordance with the decision-maker's preferences, allowing to save time in the construction of robust recommendations.

Note here that the general approach proposed in **?** does not make the study of this paper useless. In some practical situations these very general approaches might, indeed, not provide very rich conclusions, as the set of compatible outranking models can be vast when the input information from the decision-maker is poor. Furthermore, from an operational perspective, the authors of **?** also show how to determine a representative set of parameters among all compatible outranking models. They claim that this might be necessary in a real-life decision aid process to help the decision-maker to understand her preferences. We think that both approaches to stability can be seen as complementary.

Finally we would like to draw a parallel with a recent work in the value functions context **??**. The sensitivity of a solution is analyzed with respect to a partial weak order on the importance of the criteria. Assumptions on the scales of the criteria are made and linear programming techniques are used to compute the stability of the overall value of each alternative. In our case, no restriction is made on the scales of the criteria and we will consider stability degrees that can be verified without the need of linear programming.

This article is organized as follows: first, we introduce some necessary preliminary definitions, before defining the stability of outranking statements in Section 3. Then, in Section 4 we extend the concept of stability with the consideration of two additional properties under particular hypotheses. In this section we also show how this stability concept can be used in practice in a preference elicitation process.

2 Modelling outranking situations

Let $A = \{x, y, z, \ldots\}$ be a finite set of $m > 1$ potential decision alternatives evaluated on a coherent finite family $F = \{1, \ldots, n\}$ of $n > 1$ criteria. The alternatives are evaluated on ordinal or cardinal performance scales and the performance of alternative x on criterion i is denoted x_i. Without loss of generality, we assume in this article that the performances of the alternatives have to be maximized.

2.1 Marginal "at least as good as" situations

In order to measure the overall accordance with an *at least as good as* statement between any two alternatives x and y of A, with each criterion i is associated a *marginal "*at least as good as*" characteristic function* $S_i(x, y)$ whose values are defined as follows :

$$S_i(x,y) \begin{cases} = 1 & \text{if } x_i \text{ is clearly at least as good as } y_i, \\ = -1 & \text{if } x_i \text{ is clearly not at least as good as } y_i, \\ \in\,]-1;1[& \text{if it is not clear whether } x \text{ is or is not} \\ & \quad \text{at least as good as } y \text{ on criterion } i. \end{cases}$$

In the literature, this marginal characteristic function is often defined on a $[0, 1]$ scale.

Furthermore, the transition from the totally validated state (+1) to the totally non-validated state (−1) can be a linear interpolation as in the ELECTRE methods **?**, a constant function equal to the median value of the selected interval as in **??**, or it can be any monotonocially decreasing function.

2.2 Valued outranking relations

As it is classically done, we first associate with each criterion $i \in F$ a rational *weight* w_i which represents the contribution of i to the *overall support or not* of the *at least as good as* preference situation between all pairs of alternatives. Let $\mathrm{W} = (w_1, .., w_m)$ be the vector of such weights associated with F such that $0 < w_i$ ($\forall i \in F$) and let $\mathcal{W}$ be the set of such weights vectors. Then, to measure to what extent the criteria are concordant with an overall *at least as good as* statement, a *valued overall concordance index* is built via a weighted sum of the marginal concordance statements :

$$S^{\mathrm{W}}(x,y) = \sum_{i \in F} w_i \cdot S_i(x,y), \; \forall (x,y) \in A \times A.$$

In the outranking paradigm, in situations where the alternatives have very conflicting profiles, a so-called veto principle allows to invalidate to a certain extent the overall concordance index by various means **?**. If such a veto situation occurs, the concordance index is, either weakened (see the veto principle for the ELECTRE III outranking in **?**), completely invalidated (see the veto principle for the ELECTRE I outranking in **?**), or, put in an indeterminate situation **??**.

In this article, we will only deal with the second type of veto principles, which, when applied, disregards completely the considered weights vector. Consequently, as in this article we are studying the stability of the outranking relation with respect to the weights, we may without loss of generality consider here that the *overall valued outranking relation* comes down to the previously defined concordance index.

A *majority-cut* of this valued outranking index allows us to determine whether the outranking situation is validated or not. We say that alternative x *outranks* (resp. *does not outrank*) alternative y when $S^{\mathrm{W}}(x,y) > 0$, (resp. $S^{\mathrm{W}}(x,y) < 0$), *i.e.* when a weighted majority of criteria supports (resp. does not support) the marginal "*at least as good as*" preference situations between x and y. This situation is denoted $xS^{\mathrm{W}}y$ (resp. $x\not{S}^{\mathrm{W}}y$). $S^{\mathrm{W}}(x,y) = 0$ indicates a balanced situation where the criteria supporting the "*at least as good as*" preference situation between x and y are exactly as important as those who do not supporting this situation. This balanced situation is denoted $x?^{\mathrm{W}}y$. It is obvious that in another setting, where a different scale is chosen to evaluate the marginal at least as good as characteristics, this majority-cut of the outranking index is done on another level than 0 (*e.g.* if the evaluation scale of the marginal concordance is in the unit interval $[0,1]$ then the majority-cut level equals 0.5).

2.3 Weighing the performance criteria

Let $\succeq_{\mathrm{W}}$ be the preorder[1] on F associated with the natural $\geqslant$ relation on the values of the weights w_i of the vector W. $=_{\mathrm{W}}$ induces r ordered equivalence classes $\Pi_1^{\mathrm{W}} \succ_{\mathrm{W}} \ldots \succ_{\mathrm{W}} \Pi_r^{\mathrm{W}}$ $(1 \leq r \leq n)$. The criteria gathered in each equivalence class have the same weight in W and for any ranks $i < j$, those of Π_i^{W} have a higher weight than those of Π_j^{W}, the most important class being Π_1^{W}.

Definition 1 (Preorder-compatible) *Two criteria weight vectors* $\mathrm{W}, \mathrm{W}' \in \mathcal{W}$ *are said to be* preorder-compatible *if they induce the same preorder on the weights.*

Example 1 $\mathrm{W}_1 = \{2; 7; 5; 2\}$ *and* $\mathrm{W}_2 = \{3; 6; 4; 3\}$ *are preorder-compatible.*

Definition 2 (σ-preorder-compatible) *Two vectors* $\mathrm{W}, \mathrm{W}' \in \mathcal{W}$ *are said to be* σ-preorder-compatible *if* $\succeq_{\mathrm{W}'}$ *is a permutation of the equivalence classes of* $\succeq_{\mathrm{W}}$.

Let w_i and w_j (resp. w'_i and w'_j) be two components of W (resp. W'). This property of σ-preorder-compatibility can easily be verified as follows :

$$\forall i, j \in F \ : \ w_i = w_j \iff w'_i = w'_j$$

Example 2 $\mathrm{W}_1 = \{2; 2; 3; 3; 1\}$ *and* $\mathrm{W}_2 = \{4; 4; 1; 1; 2\}$ *are σ-preorder-compatible, associated with the permutation* (132).

As we shall explain later, this property may be useful when trying to consider different decision objectives, each of them gathering some equally important criteria, when the decision-maker is not able to sequence them in an order of priority.

Definition 3 (Less discriminated weights vectors) *Let* W *and* W' *in* $\mathcal{W}$ *be two weights vectors which are* not preorder-compatible. W' *is said to be* less discriminated *than* W *if its preorder* $\succeq_{\mathrm{W}'}$ *is obtained by merging some adjacent classes in* $\succeq_{\mathrm{W}}$, *i.e. when the two following conditions are verified:*

$$\begin{aligned} w_i = w_j &\implies w'_i = w'_j \quad \forall i, j \in F, \\ w_i > w_j &\implies w'_i \geqslant w'_j \quad \forall i, j \in F. \end{aligned}$$

Definition 4 (More discriminated weights vectors) *Conversely,* W' *is said to be* more discriminated *than* W *if its preorder is obtained by splitting some equivalence classes, without modifying the inequalities between the classes, i.e. iff the following condition is verified:*

$$\forall i, j \in F \ : \ w_i > w_j \implies w'_i > w'_j.$$

With these definitions in mind, we may now discuss the stability of an outranking model with respect to given criteria weights.

[1] Using classical notation, $\succ_{\mathrm{W}}$ denotes the asymmetric part of $\succeq_{\mathrm{W}}$, whereas $=_{\mathrm{W}}$ denotes its symmetric part.

3 On the stability of the outranking model

The concept of *stability* that we study in this paper characterizes, for all $(x, y) \in A \times A$, the dependence of the modelled outranking situations upon a given vector of criteria weights $\mathrm{w} \in \mathcal{W}$. This stability has originally been defined in **?**, where mathematical conditions are given for evaluating this stability. Here, we give a more intuitive expression of these properties, as well as some simplified mathematical conditions. Let x and y be two alternatives of A. $xS^{\mathrm{w}}y$ (resp. $x\not S^{\mathrm{w}}y$) is said to be:

- *Independent* (with respect to the weights): if a weighted majority of criteria supports (resp. does not support) this outranking situation, for all vectors of weights in $\mathcal{W}$;
- *Stable* (w.r.t. the weights): if a weighted majority of criteria supports (resp. does not support) the outranking situation between x and y for any vector of weights which are preorder-compatible with w. This situation only depends on the preorder of w, and not its precise numerical values;
- *Unstable* (w.r.t. the weights): if a weighted majority of criteria supports (resp. does not support) this outranking situation for w, but not for every vector of weights which is preorder-compatible with w. This situation therefore essentially depends on the accuracy of the numerical value given to each criterion weight.

Example 3 *Let us illustrate our discourse with an example concerning 4 alternatives evaluated on 9 criteria. To simplify the explanations and without loss of generality, we are not considering here any indifference or preference discrimination threshold which could be used in the marginal concordance index. A vector of weights* $\mathrm{w} = \{3, 3, 3, 2, 2, 2, 1, 1, 1\}$, *inducing the importance ordering* $\{g_1, g_2, g_3\} \succ_{\mathrm{w}} \{g_4, g_5, g_6\} \succ_{\mathrm{w}} \{g_7, g_8, g_9\}$, *is defined and the performance table, on which all the evaluations have to be maximized, is given in the left part of Table 1. The valued outranking relation is given in the right part of this table, with values normalized between -1 and 1 to emphasize the closeness to the median and extremal values.*

Table 1: Performance table and associated valued outranking relation

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	S^{w}			
w	3	3	3	2	2	2	1	1	1	a	b	c	d
a	5	2	6	4	1	3	7	6	5		-0.12	-0.22	-0.12
b	3	4	5	7	2	6	4	5	8	0.12		-0.12	-1.00
c	6	6	3	5	8	2	5	7	3	0.22	0.12		-0.33
d	4	7	6	8	6	7	6	6	9	0.56	1.00	0.33	

We can identify two issues for which the concept of stability proposes an answer. First, how reliable are these outranking values, knowing that I am not very confident in the precise values of the weights, but I am sure about their preorder? Second, considering a non-well determined value of the outranking *relation (i.e. close to 0 in this setting), can this situation be considered as faithfully reflecting the decision-maker's mind or is it just an incidental situation created by a not fine-enough tuning of the weights?*

In the following subsections, we detail the previously introduced levels of stability and give simple formulas for testing their validity.

3.1 Independence from criteria weights

An outranking situation is de facto *independent* from every vector of weights when the first alternative dominates or is dominated by the second one, which is the case when at least one criterion validates (or invalidates) and no criterion invalidates (or validates) the "at least as good as" situation.

Proposition 1 (Independence)

$$\text{“}xS^{\mathrm{w}}y\text{” is independent} \iff \begin{cases} \forall i \in F : S_i(x,y) = 1 \text{ or } S_i(x,y) = 0\,; \\ \exists i \in F : S_i(x,y) = 1. \end{cases} \tag{1}$$

$$\text{“}x\not{S}^{\mathrm{w}}y\text{” is independent} \iff \begin{cases} \forall i \in F : S_i(x,y) = -1 \text{ or } S_i(x,y) = 0\,; \\ \exists i \in F : S_i(x,y) = -1. \end{cases} \tag{2}$$

$$\text{“}x?^{\mathrm{w}}y\text{” is independent} \iff \forall i \in F : S_i(x,y) = 0. \tag{3}$$

Proof. All criteria weights w_i being strictly positive by definition, $S^{\mathrm{w}}(x,y) = \sum_i S_i(x,y)\cdot w_i$ will always be positive, negative, respectively zero, independently of any numerical setting of the weights.

Example 4 *Back to our example, we easily verify that alternative d is at least as good as b on each of the criteria. In that case, d outranks b independently of any vector of weights. As no performance discrimination thresholds are considered in this example, d will also be considered as strictly preferred to b, independently of any weights vector.*

3.2 Stability with preorder-compatible weights

Let $c^{\mathrm{w}}_k(x,y)$ be the sum of "*at least as good as*" characteristics $S_i(x,y)$ for all criteria $i \in \Pi^{\mathrm{w}}_k$. Furthermore, let $C^{\mathrm{w}}_k(x,y) = \sum_{i=1}^{k} c^{\mathrm{w}}_i(x,y)$ be the cumulative sum of "*at least as good as*" characteristics for all criteria having importance at least equal to the one associated with Π^{w}_k, for all k in $\{1,\ldots,r\}$. Intuitively speaking, it is the set of the most important criteria, on which we may limit the decision if the other ones are insignificant.

The following conditions characterize the stability of outranking situations (see also **?** for similar conditions in an additive value functions context):

Proposition 2 (Stability)

$$\text{“}xS^{\mathrm{w}}y\text{” is stable} \iff \begin{cases} \forall k \in 1,\ldots,r : C^{\mathrm{w}}_k(x,y) \geqslant 0\,; \\ \exists k \in 1,\ldots,r : C^{\mathrm{w}}_k(x,y) > 0. \end{cases} \tag{4}$$

$$\text{“}x\not{S}^{\text{w}}y\text{” is stable} \iff \begin{cases} \forall k \in 1,\ldots,r : C_k^{\text{w}}(x,y) \leqslant 0\,; \\ \exists k \in 1,\ldots,r : C_k^{\text{w}}(x,y) < 0. \end{cases} \tag{5}$$

$$\text{“}x?^{\text{w}}y\text{” is stable} \iff \forall k \in 1,\ldots,r : C_k^{\text{w}}(x,y) = 0. \tag{6}$$

Proof. We prove Equivalence (4), by showing that the right-hand condition is, indeed, necessary and sufficient.

First, let us assume that $\forall k \in 1,\ldots,r : C_k^{\text{w}}(x,y) \geqslant 0$ and also that $\exists k \in 1,\ldots,r : C_k^{\text{w}}(x,y) > 0$. It is easy to verify that:

$$\begin{array}{rcl}
S^{\text{w}}(x,y) & = & \displaystyle\sum_{k=1}^{r} c_k^{\text{w}}(x,y)\cdot w_k \quad = \quad c_1^{\text{w}}(x,y)\cdot w_1 + c_2^{\text{w}}(x,y)\cdot w_2 + \ldots + c_r^{\text{w}}(x,y)\cdot w_r \\
& = & \boldsymbol{C}_1^{\text{w}}(x,y)\cdot w_1 + c_2^{\text{w}}(x,y)\cdot w_2 + \ldots + c_r^{\text{w}}(x,y)\cdot w_r \\
& & \text{As } w_2 < w_1 \text{ and } C_1^{\text{w}}(x,y) \geqslant 0\text{:} \\
& \geqslant & \big(C_1^{\text{w}}(x,y)\cdot \boldsymbol{w_2} + c_2^{\text{w}}(x,y)\cdot w_2\big) + \ldots + c_r^{\text{w}}(x,y)\cdot w_r \\
& \geqslant & C_{\boldsymbol{2}}^{\text{w}}(x,y)\cdot w_2 + c_3^{\text{w}}(x,y)\cdot w_3 + \ldots + c_r^{\text{w}}(x,y)\cdot w_r \\
& & \text{As } w_3 < w_2,\ C_2^{\text{w}}(x,y) \geqslant 0 \text{ and } C_2^{\text{w}}(x,y) + c_3^{\text{w}}(x,y) = C_3^{\text{w}}(x,y)\text{:} \\
& \geqslant & C_{\boldsymbol{3}}^{\text{w}}(x,y)\cdot w_3 + c_4^{\text{w}}(x,y)\cdot w_4 + \ldots + c_r^{\text{w}}(x,y)\cdot w_r \\
& [..] & \\
& \geqslant & C_r^{\text{w}}(x,y)\cdot w_r \\
S^{\text{w}}(x,y) & \geqslant & 0
\end{array}$$

As there exists at least one strictly positive cumulative sum, the inequation is strict. Hence, $S^{\text{w}}(x,y) > 0$ and the right-hand condition in Equivalence (4) is therefore sufficient.

Conversely, let us now assume that there exists a strictly negative cumulative sum $C_l^{\text{w}}(x,y) < 0$. For the ease of the proof, let us furthermore assume that this is the only one, *i.e.* $\forall k \neq l : C_k^{\text{w}}(x,y) \geqslant 0$. If we show, indeed, that it is possible to find a vector of weights, which is compatible with the preorder, and which invalidates the outranking situation, then it will be even more possible to do so when there are more such strictly negative cumulative sums. Now, we face two exclusive cases: either, $l = 1$, which means that the most important class has more criteria against the validation than in favor, or, $l > 1$. In the first case, it is sufficient to associate a very large weight with the first class and a very small for all other classes in order to obtain a negative outranking value. In the second case, let us define $w_1 = 1 + \frac{-C_l^{\text{w}}(x,y)}{\sum_{t\neq l} c_t}$, the weight associated with the most important class, $w_l = 1$ and $w_{l+1} = \frac{-C_l^{\text{w}}(x,y)}{\sum_{t\neq l} c_t}$. We then compute the outranking value:

$$\begin{array}{rcl}
S^{\text{w}}(x,y) & = & \displaystyle\sum_{k=1}^{r} c_k^{\text{w}}(x,y)\cdot w_k \\
& & \text{As } \displaystyle\sum_{k=1}^{l-1} c_k^{\text{w}}(x,y) = C_{l-1}^{\text{w}}(x,y) \geqslant 0 \text{ and } \forall k = 2..l-1,\ w_k < w_1 \text{ and} \\
& & \text{as } \displaystyle\sum_{k=l+1}^{r} c_k^{\text{w}}(x,y) = C_k^{\text{w}}(x,y) - C_{l-1}^{\text{w}}(x,y) > 0 \text{ and } \forall k = l+2..r,\ w_k < w_{l+1}\ : \\
S^{\text{w}}(x,y) & < & C_{l-1}^{\text{w}}(x,y)\cdot \boldsymbol{w_1} + c_l^{\text{w}}(x,y)\cdot w_l + \displaystyle\sum_{k=l+1}^{r} c_k^{\text{w}}(x,y)\cdot \boldsymbol{w_{l+1}}
\end{array}$$

$$
\begin{array}{rcl}
 & < & C^{\mathrm{W}}_{l-1}(x,y)\cdot\Big(1+\dfrac{-C^{\mathrm{W}}_{l}(x,y)}{\sum_{t\neq l} c_t}\Big)+c^{\mathrm{W}}_{l}(x,y)+\sum_{k=l+1}^{r} c^{\mathrm{W}}_{k}(x,y)\cdot\Big(\dfrac{-C^{\mathrm{W}}_{l}(x,y)}{\sum_{t\neq l} c_t}\Big) \\
 & < & C^{\mathrm{W}}_{l}(x,y)+\dfrac{-C^{\mathrm{W}}_{l}(x,y)\cdot\sum_{t\neq l} c_t}{\sum_{t\neq l} c_t} \quad = \quad C^{\mathrm{W}}_{l}(x,y)-C^{\mathrm{W}}_{l}(x,y) \\
S^{\mathrm{W}}(x,y) & < & 0
\end{array}
$$

The right-hand condition of Equivalence (4) is, hence, also a necessary one. Proof of Equivalence ((5)) is similar when inverting criteria in favor and in disfavor. Finally, Equivalence ((6)) is obvious.

3.3 Residual instability

Any outranking situation that does not verify the stability conditions above is said to be *unstable*. In this latter case, when W validates (resp. invalidates) the outranking, it is possible to find weights vectors which are preorder-compatible with W and which invalidate (resp. validate) it or which may generate a balanced situation. A precise and accurate setting of the individual criteria weights becomes, hence, required in order to remove this ambiguity. If this cannot be achieved, the reliability of such an outranking situation will be weak. This will be even more be the case, when the associated outranking value $S^{\mathrm{W}}(x,y)$ is weakly determined (close to 0 in our valuation).

Table 2: Computation of the stability of some outranking situations

Proposition	c^{W}_1	c^{W}_2	c^{W}_3	C^{W}_1	C^{W}_2	C^{W}_3	Denotation
$bS^{\mathrm{W}}a$	-1	3	-1	-1	2	1	unstable
$cS^{\mathrm{W}}b$	1	-1	1	1	0	1	stable
$a\not S^{\mathrm{W}}b$	1	-3	1	1	2	-1	unstable
$c\not S^{\mathrm{W}}d$	-1	-1	-1	-1	-2	-3	stable

	a	b	c	d
a		−U	−S	−U
b	+U		−S	−I
c	+S	+S		−S
d	+S	+I	+S	

I: Independent outranking statement; S: Stable; U: Unstable.
+ (resp. −): Positive (resp. negative) outranking statement.

Example 5 *Back to our example, we now compute, in the left part of Table 2, the stability of some of the previous outranking situations. We can see that* $S^{\mathrm{W}}(b,a) = S^{\mathrm{W}}(c,b) = 0.12$*, but these two situations have quite different behaviors in terms of stability. Indeed, when looking at the computation details in Table 2, "*$cS^{\mathrm{W}}b$*" is stable, whereas "*$bS^{\mathrm{W}}a$*" is not. The weak value associated with "*$cS^{\mathrm{W}}b$*" is thus not a weakly-determined one, contrary to the second relation "*$bS^{\mathrm{W}}a$*"[2]. Notice that a sensitivity analysis that is not taking into account the preorder of* W *may have considered both relations as potentially doubtful. Furthermore, considering wrongly that an outranking relation is clearly validated only if it is associated with a clear positive value, one may be tempted to doubt both situations. However, as we have shown, the first one is not, within the limits of the given preorder of weights, incidental to any precise setting of the weights.*

[2] The careful reader shall easily verify that, considering $\mathrm{W}' = (6,6,6,2,2,2,1,1,1)$, or $\mathrm{W}^* = (8,8,8,3,3,3,2,2,2)$, both preorder-compatible with W, we obtain $b\not S^{\mathrm{W}'}a$ and $aS^{\mathrm{W}^*}b$.

Assuming an explicit validation of the preorder $\succeq_{\mathrm{W}}$, it is then clearly legitimate to consider a stable situation, even if it is not-well determined, as implicitly validated, whereas an unstable and not-well determined situation has to be explicitly validated by the decision-maker. As it is not possible to ask her to validate the complete set of outranking statements, the concept of stability allows her to focus on sensitive outranking situations only, decreasing the time of the validation protocol and increasing her confidence in the final outranking digraph. As a result, any post-exploitation of the outranking digraph will be more robust.

4 Refining the outranking stability levels and robust elicitation of the weights

In this section we first define two additional stability levels which allow to characterize more precisely a stable outranking situation. Then we outline an elicitation protocol to determine robust weights. But first we introduce an important limitation for the stability of an outranking model.

4.1 Stable with less discriminated weights

An important property for our purpose is described in **?**. Let $\mathrm{W}_{\mathbf{1}}$ be the weights vector for which all the criteria weights equal 1. Then:

Property 1 (Limitation of the stability ?)

$$x \not S^{\mathrm{W}_{\mathbf{1}}} y \quad \Longrightarrow \quad \nexists \mathrm{W} \in \mathcal{W},\ \textit{s.t. } xS^{\mathrm{W}}y \textit{ is stable} \tag{7}$$

$$x S^{\mathrm{W}_{\mathbf{1}}} y \quad \Longrightarrow \quad \nexists \mathrm{W} \in \mathcal{W},\ \textit{s.t. } x\not S^{\mathrm{W}}y \textit{ is stable} \tag{8}$$

In other words, when more than half of the criteria does not validate (resp. validates) an outranking situation, it is impossible to find a vector of criteria weights that validates (resp. does not validate) this situation in a stable manner.

Proof. In the verification of the stability of an outranking statement, the cumulated sum $C_r^{\mathrm{W}}(x,y)$, associated with the weakest importance classes of any vector of weights W, is by construction always equal to the sum of all the marginal concordances, *i.e.* $C_r^{\mathrm{W}}(x,y) = S^{\mathrm{W}_{\mathbf{1}}}(x,y)$, for all W. If we assume that $xS^{\mathrm{W}_{\mathbf{1}}}y$, namely $C_r^{\mathrm{W}}(x,y) = S^{\mathrm{W}_{\mathbf{1}}}(x,y) > 0$, there is at least one strictly positive cumulated sum, wich is incompatible with the verification of a stable negative outranking statement (see inequation (5) of Proposition 2). We prove the second statement of the property above similarly. □

Property 1 emphasizes the importance of the vector of equal weights, as the corresponding outranking digraph will be automatically completely stable. The stability limitation highlights, moreover, the noteworthy fact that we cannot rely solely on the set of stable statements for making useful decision recommendations, as they will always tend to agree with those obtained with equi-important weights.

If we make a comparison between the concepts of stability and the *necessary* and *possible* outranking statements (see **?**), we can see that a stable outranking statement is also a necessary one, assuming that the given preferential information is the complete preorder on the weights (*i.e.* the set of ordinal constraints between the criteria weights). From the previous property we then deduce that it is impossible to find a necessary outranking statement which is not in accordance with the one obtained with $\mathrm{W}_{\mathbf{1}}$, if the only preferential information which is considered are the ordinal constraints between the criteria weights. The proof is obvious, as the complete preorder is the largest set of such preferential information that can be constructed.

This important property will be useful in the construction of a progressive method for a robust elicitation of the weights, in Section 4.4.

4.2 Stable under permuted weight equivalence classes

Let us now assume a different situation, where the criteria have been gathered under some more general objectives that a decision-maker does not want to order according to their importance, but where under a given objective the criteria have equal weights.

Definition 5 (σ-stability) *A positive (resp. negative) outranking situation $xS^{\mathrm{W}}y$ (resp. $x\not S^{\mathrm{W}}y$) is said to be* σ-stable *(w.r.t.* W*) when a weighted majority of criteria supports (resp. does not support) the situation between x and y for any vector of weights σ-preorder-compatible with* W.

The following proposition gives us a test for the σ-stability of any outranking statement:

Proposition 3 (σ-stability)

$$\text{“}xS^{\mathrm{W}}y\text{” } \textit{is } \sigma\textit{-stable} \iff \begin{cases} \forall k \in 1,\dots,r : c_k^{\mathrm{W}}(x,y) \geqslant 0\,; \\ \exists k \in 1,\dots,r : c_k^{\mathrm{W}}(x,y) > 0. \end{cases} \tag{9}$$

$$\text{“}x\not S^{\mathrm{W}}y\text{” } \textit{is } \sigma\textit{-stable} \iff \begin{cases} \forall k \in 1,\dots,r : c_k^{\mathrm{W}}(x,y) \leqslant 0\,; \\ \exists k \in 1,\dots,r : c_k^{\mathrm{W}}(x,y) < 0. \end{cases} \tag{10}$$

$$\text{“}x?^{\mathrm{W}}y\text{” } \textit{is } \sigma\textit{-stable} \iff \forall k \in 1,\dots,r : c_k^{\mathrm{W}}(x,y) = 0. \tag{11}$$

Proof. As we study all the possible permutations between the equivalence classes, each class can be considered as the most important one. As for all W and (x,y), $C_1^{\mathrm{W}}(x,y) = c_1^{\mathrm{W}}(x,y)$, at least the most important class of any preorder has to verify Proposition 2, *i.e.* that $c_1^{\mathrm{W}}(x,y) \geqslant 0$. According to the permutations, all the equivalence classes have to verify the inequality. The condition is then a necessary one. The other way around, assuming that all the

Table 3: σ-stability

Testing some couples of alternatives

Proposition	c_1^{w}	c_2^{w}	c_3^{w}	σ-stability
$cS^{\mathrm{w}}b$	1	-1	1	$\times$
$dS^{\mathrm{w}}a$	1	3	1	σ-stable
$a\not S^{\mathrm{w}}d$	1	-3	1	$\times$
$c\not S^{\mathrm{w}}d$	-1	-1	-1	σ-stable

Complete relation

	a	b	c	d
a		$-$	$-$	$-$
b	$+$		$-$	$-\sigma$
c	$+$	$+$		$-\sigma$
d	$+\sigma$	$+\sigma$	$+\sigma$	

$+/-$: Positive/Negative outranking statement
σ: σ-stable outranking statement

$c_k^{\mathrm{w}}(x,y)$ are greater or equal to 0, no matter the order we have between the classes, any cumulative sum $C_k^{\mathrm{w}}(x,y)$ will remain greater or equal to 0. Hence, the condition is sufficient and we obtain the equivalence. $\square$

The σ-stability level corresponds in fact to the verification of a group unanimity condition. Assuming that an outranking situation is granted for each equivalence class, no matter the relative importance of each class, this situation will always be granted. Consequently such an outranking a situation is highly reliable.

Example 6 *Let us now assume that our example is modeling the evaluations of a jury composed of three judges who evaluated a set of candidates, based on three criteria. The judges are unable to agree on the way to prioritise these criteria, but agree on the fact that the importance of each judge on each criterion should be the same. Consequently, we group the evaluations in three classes (based on the fact that they concern the same criterion), arbitrarily assign the weights 1, 2 and 3 to the classes as in Table 1, and compute the σ-stability property in the right part of Table 3.*

We easily observe that d outranks all the other alternatives without the need to order the importance classes. Moreover, d is clearly preferred to b and c (as they will never outrank d under the working hypotheses). However, as a might outrank d, they both could be considered as indifferent. Again, without further information, we cannot rely on the statements that are not σ-stable. Nevertheless, considering for instance a best choice problematic, we could rationally recommend d as the best alternative, but an in depth discussion on the importance of the criteria will be required to rank the other alternatives.

4.3 Stable with more discriminated weights

Although the computation of the stability eases the validation of the stable outranking statements, there is a strong hypothesis on the fact that two criteria, in the same weight equivalence class, must be given exactly the same numerical weight. It is not difficult to imagine a situation where the decision-maker gathers criteria in the same equivalence class, while still having a doubt about the fact that these criteria do have exactly the same importance in the pairwise comparisons.

Definition 6 (Extensible stability) *A positive (resp. negative) outranking situation $xS^{\mathrm{W}}y$ (resp. $x\not S^{\mathrm{W}}y$) is said to be* extensibly stable *(w.r.t.* W*) when a weighted majority of criteria supports (resp. does not support) the situation between x and y for any vector of weights more discriminated than* W.

For any pair of alternatives (x, y), let us define $\mathrm{W}^{\mp}$ as the vector of weights associated with the preorder $\Pi^{\mathrm{W}^{\mp}}(x, y)$ which is defined as follows:

$$\Pi^{\mathrm{W}^{\mp}}(x, y) = \Pi_1^{\mathrm{W}-}(x, y) \succ \Pi_1^{\mathrm{W}+}(x, y) \succ \ldots \succ \Pi_r^{\mathrm{W}-}(x, y) \succ \Pi_r^{\mathrm{W}+}(x, y)$$

$\Pi^{\mathrm{W}^{\mp}}(x, y)$ is in fact obtained by separating the importance classes of Π^{W} between the criteria against and the ones in favor of the situation. This preorder is the worst case we can create from a preorder which is more discriminated than Π^{W}, when considering the validation of an outranking situation. Indeed, the criteria in favor (resp. against) are the least (resp. most) possibly important. In a similar way, we also define $\mathrm{W}^{\pm}$ as the worst case when trying to invalidate an outranking situation, splitting each equivalence class and prioritizing the criteria in favor of the validation. It follows that:

Proposition 4 (Extensible stability)

$$\begin{array}{ccc} \text{“}xS^{\mathrm{W}^{\mp}}y\text{” is stable} & \Longleftrightarrow & \text{“}xS^{\mathrm{W}}y\text{” is extensibly stable} \\ \text{“}x\not S^{\mathrm{W}^{\pm}}y\text{” is stable} & \Longleftrightarrow & \text{“}x\not S^{\mathrm{W}}y\text{” is extensibly stable} \end{array}$$

Proof. To prove the first equivalence, if we assume that “$xS^{\mathrm{W}^{\mp}}y$” is stable, it is easy to verify that any vector of weights W' more discriminated than W and different from $\mathrm{W}^{\mp}$ will contain either a criterion in favor of the outranking situation with a higher weight than in $\mathrm{W}^{\mp}$ or a criterion in disfavor with a lower weight than in $\mathrm{W}^{\mp}$. Then, “$xS^{\mathrm{W}'}y$” will be also stable. The other way around, if we assume that “$xS^{\mathrm{W}}y$” is extensibly stable, then it is stable for every vector of weights more discriminated than W, especially $\mathrm{W}^{\mp}$.

The second equivalence is verified in a similar manner. □

A careful reader may notice that we are not giving any condition for testing the extensible stability of a balanced situation. In fact, we have the property that only the balanced situations that are independent of the weights can be extensibly stable. This is easily proved, when supposing an extensibly stable balanced situation, which means that we can refine each equivalence class without modifying the balanced situation. But, if there exist criteria for which $S_i(x, y) \neq 0$, it automatically means that they are compensated by other criteria in the same class. Splitting the equivalence class will therefore result in a disruption of the balance. Consequently, the only extensibly stable balanced relations are relations where $S_i(x, y) = 0$ for all criteria, which are independent from any vector of weights.

Example 7 *Returning to our example, let us suppose that the decision-maker did not provide a precise preorder, but only grouped the criteria according to*

Table 4: Extensible stability test for the proposition "$dS^{\mathrm{w}}a$"

$\Pi^{\mathrm{w}} = \{g_2\} \succ \{g_1, g_3\} \succ \{g_4, g_5, g_6\} \succ \{g_7, g_8, g_9\}$

$\Pi^{\mathrm{w}\mp}$	$\Pi_1^{\mathrm{w}-}$ $\emptyset$	$\Pi_1^{\mathrm{w}+}$ $\{g_2\}$	$\Pi_2^{\mathrm{w}-}$ $\{g_1\}$	$\Pi_2^{\mathrm{w}+}$ $\{g_3\}$	$\Pi_3^{\mathrm{w}-}$ $\emptyset$	$\Pi_3^{\mathrm{w}+}$ $\{g_4, g_5, g_6\}$	$\Pi_4^{\mathrm{w}-}$ $\{g_7\}$	$\Pi_4^{\mathrm{w}+}$ $\{g_8, g_9\}$
$c_k^{\mathrm{w}\mp}(d, a)$	0	1	-1	1	0	3	-1	2
$C_k^{\mathrm{w}\mp}(d, a)$	0	1	0	1	1	4	3	5

Table 5: Extensible stability

$\{g_1, g_2, g_3\} \succ \{g_4, g_5, g_6\} \succ \{g_7, g_8, g_9\}$

	a	b	c	d
a		$-$	$-$	$-$
b	$+$		$-$	$-$ES
c	$+$	$+$		$-$
d	$+$	$+$ES	$+$	

$\{g_2\} \succ \{g_1, g_3\} \succ \{g_4, g_5, g_6\} \succ \{g_7, g_8, g_9\}$

	a	b	c	d
a		$-$	$-$ES	$-$
b	$+$		$-$	$-$ES
c	$+$ES	$+$		$-$ES
d	$+$ES	$+$ES	$+$ES	

$+/-$: Positive/Negative outranking statement

ES: extensibly stable outranking statement

whether he considers them as very important (g_1, g_2, g_3), important (g_4, g_5, g_6) or less important (g_7, g_8, g_9) in the pairwise comparisons. We associate some weights from 1 to 3 to the criteria, according to the initial preorder and compute the extensible stability relation in left part of Table 5 (An example of how to compute the extensible stability property is given in Table 4). At this stage, we can observe that there are many uncertainties on the outranking statements. If the decision-maker is unable to discriminate better the preorder, the exploitation result which can be obtained from the current outranking relation will have a very low degree of reliability.

Suppose that later in the discussion, the decision-maker acknowledges that criterion g_2 is clearly the most important one. The new outranking relation and the associated extensible stability property are given in the right part of Table 5. We easily see that the number of extensibly stable relations has increased, reducing consequently the number of validations that the decision-maker has to make on the unstable outranking statements. Notice that if the decision-maker is certain about the given preorder, namely that two criteria with the same weights have exactly the same importance, the exploitation of the outranking graph can be performed right away.

Let us finally note that this progressive setting of precise criteria weights can be useful when facing multiple decision-makers which agree with a basic preorder on the weights of the criteria, and who wish to refine it in different manners. The extensible stability will highlight the conflictual situations and those that are not so.

4.4 A progressive method for eliciting robuster criteria weights

As we mentioned when presenting Property 1 on the limitation of the stability concept, taking into account solely the preferential information expressed by ordinal constraints on the weights of the criteria (whether complete as in our case or partial) is not sufficient for producing robust decision recommendations. It could, therefore, be tempting to obtain preciser criteria weights from the decision-maker by trying to enhance the effective discrimination between more or less equally weighted criteria. It therefore becomes interesting to complete these ordinal constraints on the criteria weights with some more precise preferential information like outrankings directly validated by the decision-maker or, even more precise, numerical ratios between some weights.

Consequently, the progressive direct elicitation protocol for criteria weights, which we propose here and which relies on the concept of stability to capture the weights of the considered criteria, is divided into three stages:

1. The decision-maker defines an initial preorder on the weights of the criteria with a limited number of equi-importance classes.
2. This preorder is progressively refined, until a complete and clear final criteria weights preorder may be valiadated by the deicison-maker..
3. The remaining unstable outranking situations are individually inspected in order to find the precise compatible vector of criteria weights validating, resp. invalidating, them.

In the first stage, the decision-maker is asked to sort the set of criteria into a very small set of ordered rough classes, according to their weights in her opinion. For instance, we can consider the sorting of the criteria into three classes: the "very important" criteria, the "important" ones and the "less important" ones. Once this initial preorder is validated, all the outranking statements that validate the extensible stability property are implicitly validated.

In the second stage, the goal is to refine the initial preorder in order to obtain a preorder which is better in accordance with the decision-maker's preferences. Two approaches can be considered for this stage, each of them benefiting from the stability principles defined in this article. On the one hand, a direct approach, where the decision-maker adds the discrimination by herself, and on the other hand, an indirect approach, where the decision-maker has to validate or invalidate certain outranking situations, from which we deduce the necessary modifications on the preorder of the weights.

If the direct approach is chosen in this stage, the stability can be used to test various hypotheses on the relative importance of the criteria. It allows to study the evolution of the stability at each refinement, by showing the decision-maker the outranking situations which become unstable.

If the indirect option is however chosen at this stage, the stability avoids to question the decision-maker on outranking situations which verify the extensible stability property. This indirect approach also allows to determine weights which maximizes the stability over the whole graph : the weights vector is chosen in order to minimize its influence on the outranking digraph, to ease the discussion with the decision-maker and to avoid unfounded algorithmic choices. To illustrate this, the interested reader can refer to **?**.

Once the preorder has been clearly validated, all the stable outranking statements can be considered as highly reliable, even when associated with a weak outranking value. In this last stage, the weights have then to be tuned precisely, in order to fix correctly the remaining unstable outranking statements. Again, this can be done directly, but it supposes a quite explicit knowledge of the relative importance of the criteria. Similarly as in stage 2, an indirect approach can be used to discuss some unstable outranking statements with the decision-maker.

5 Conclusion and future works

Considering the stability of majority-cut outranking relations gives an efficient tool for understanding the dependencies of the outranking digraph with respect to the precisely chosen criteria weights. It may lead, both, to a stabler validation of outranking situations by highlighting the assumptions induced by the outranking model parameters, and, a more solid exploitation of the outranking for the selecting, rnking or sorting problematique. The impact of an imprecise fixation of the criteria weights may thus be measured, or even limited within the context of an ordinal regression when maximizing the stability of the digraph **?**.

Consequently, it may efficiently improve the decision-maker's confidence in the final decision recommendation, as the person is certainly more comfortable in validating a preorder than a precise numerical vector of weights. An important application of the majority-cut stability concept appears, thus, in the sorting of alternatives, as for instance in Electre Tri models **??**. Indeed, a decision-maker will certainly be more satisfied with a sorting result when all the outranking situations which need to be considered in the assignment rules appear to be stable with a given criteria weights preorder. Some future work will be dedicated to this issue, and more specifically to the elicitation of category delimiting profiles which maximize the stability of the sorting result when based on assignment examples given by the decision-maker.

Finally, the majority-cut stability concept opens, more generally, new perspectives in the definition of robust elicitation protocols by efficiently discriminating between stably validated or invalidated outranking situations and those pairs of alternatives which require a specific attention by the decision-maker. This knowledge may definitely decrease the operational complexity of future preference elicitation protocols.